Semisimple Complex Lie Algebras

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ABSTRACT
This paper examines Lie algebras with particular focus on finite-dimensional Lie algebras over $\mathbb{C}$, building up to the classification of semisimple complex Lie algebras. We draw heavily from Erdmann and Wildon's book *Introduction to Lie Algebras* [3]. After some brief historical background, we introduce properties like solvability and semisimplicity, define the classical matrix Lie algebras, then give a whirldwind tour of the classification by root systems and Dynkin diagrams.The appendix includes solutions to many exercises from [3].

1 Historical notes

In the 1862-1863, Sophus Lie (1842-1899) attended a series of lectures by Sylow on Galois theory while studying at the University of Oslo [6]. This may have been the kernel of Lie's great inspiration: to look at differential equations in the way that Galois looked at polynomial equations, that is, by considering symmetry groups. First known as “continuous groups” or “infinitesimal groups,” these symmetry groups are now known as Lie groups [2]. In his 1874 paper *Zur Theorie des Integrabilitetsfaktors*, Lie demonstrated the validity of his intuition with a theorem relating the stability group of a differential equation to its solvability via integration.

Wilhelm Killing (1847-1923) began studying the closely related structures now known as Lie algebras from a different starting point than Lie. While Lie began with differential equations, Killing had a geometrical interest in “the problem of classifying infinitesimal motions of a rigid body in any type of space” [2]. In the second of a series of four papers published in 1888-1890 called *Die Zusammensetzung der stetigen, endlichen Transformationsgruppen*, Killing gave a full classification of simple complex Lie algebras. In this paper, he introduced what are now known as Cartan subalgebras, Cartan matrices, and root systems [2].

In his doctoral thesis, Élie Cartan (1869-1951) extended Killing's work on the classification of simple complex Lie algebras. Depending on whom one believes, Cartan either fixed a few minor gaps or did a major repair job. Coleman [2] says that “the first two thirds” of Cartan's thesis is “essentially a commentary” on Killing's paper, while Hawkins [4] describes Killing's paper as “imperfect work” in a “tentative form,” and refers to “Cartan's successful and brilliant reworking of Killing's theory.”

Regardless, it is certain that Killing's paper lacked some amount of details and rigor, and Cartan's thesis gave a fuller presentation of the classification. In his thesis, Cartan introduced what is now called the Killing form, as well as his “criterion for solvability” and “criterion for semisimplicity.” Cartan went on to do more work in the classification of Lie algebras, including his classification of simple real Lie algebras.

2 Properties of Lie Algebras

We assume that the reader is familiar with the linear algebra concepts of vector spaces, linear maps, and representing linear maps by matrices.

Definition 2.1. *A **Lie algebra** is a vector space L over a field F with a bilinear bracket $[,] : L \times L \to L$ that satisfies $[x, x] = 0$ and $[x, [y, z]] + [y, [z, x]] + [z, [y, x]] = 0$ for all $x, y, z \in L$.*

Specifically, the bilinear property of the bracket is equivalent to the following:

$$\left[\sum_i a^i x_i, \sum_j b^j y_j\right] = \sum_{i,j} a^i b^j [x_i, y_j]$$

Proposition 2.2. *Let L be a Lie algebra. Then $[x, y] = -[y, x]$ for $x, y \in L$.*

Proof. See [3] page 1. □

This paper is mostly concerned with finite-dimensional Lie algebras over the fields $\mathbb{R}$ and $\mathbb{C}$, so frequently finite-dimensionality is taken as an unstated assumption. Working in finite dimensions simplifies things because it allows us to always represent a linear map by a matrix. The field can be left unspecified for many results, but many of the later theorems restrict to the case where the field is $\mathbb{C}$.

Lie algebras have many analogous concepts to groups and rings, including subalgebras (corresponding to subgroups), ideals, being abelian, center (denoted $Z(L)$), homomorphisms, isomorphisms (denoted $\cong$), and quotient algebras. For definitions of these, see chapters 1 and 2 of [3]. As just one example of the analogy between Lie algebras and rings, take the following proposition.

Proposition 2.3 (Exercise 1.6[1])**.** *Let L_1, L_2 be Lie algebras and let $\phi : L_1 \to L_2$ be a Lie algebra homomorphism. Then $\ker\phi$ is an ideal of L_1 and $\operatorname{im}\phi$ is a subalgebra of L_2.*

Proof. First we show that $\ker\phi$ is an ideal of L_1. Let $x \in L_1, y \in \ker\phi$. Then

$$\phi([x, y]) = [\phi(x), \phi(y)] = [\phi(x), 0] = 0 \implies [x, y] \in \ker\phi$$

Now we show that $\operatorname{im}\phi$ is a subalgebra of L_2. Let $x, y \in \operatorname{im}\phi$. Then there exist $x', y' \in L_1$ such that $\phi(x') = x$ and $\phi(y') = y$. Then

$$[x', y'] \in L_1 \implies [x, y] = [\phi(x'), \phi(y')] = \phi([x', y']) \in \operatorname{im}\phi$$

□

The above proposition is exactly parallel to one about rings: the kernel of a ring homomorphism is an ideal of the domain, and the image is a subring of the codomain.

[1] All exercise numbers refer to Erdmann and Wildon, *Introduction to Lie Algebras.*

2.1 Solvable and Nilpotent

Because of Proposition 2.3 above, we know that Lie algebra isomorphisms must preserve properties related to ideals, so constructions involving ideals are central to classifying Lie algebras.

Definition 2.4. *Let L be a Lie algebra with ideals I, J. We define the bracket of I, J by*

$$[I, J] = \operatorname{span}\{[x, y] : x \in I, y \in J\}$$

Proposition 2.5. *Let L be a Lie algebra with ideals I, J. Then $[I, J]$ is an ideal of L.*

Proof. See [3] page 12. □

In particular, we use the symbol L' or $L^{(1)}$ to refer to $[L, L]$, which is called the **derived algebra** of L. We can also consider the derived algebra of L', which is an ideal of L', and so on. We refer to the kth derived algebra by $L^{(k)}$, so we have the **derived series** of ideals of L:

$$L \supset L^{(1)} \supset L^{(2)} \supset L^{(3)} \supset \dots$$

Definition 2.6. *A Lie algebra L is* ***solvable*** *if $L^{(k)} = 0$ for some $k \geq 1$.*

Note that the symbol "0" here refers not to the additive identity of the vector space L, but to the singleton set containing that element. This common abuse of notation is unfortunate, since "0" is also used for the zero vector in L and the zero element of the underlying field F.

The adjective "solvable" is applied to both Lie algebras and to groups, and the parallel usage is not coincidental. The next two lemmas indicate how a requirement analogous to the definition of solvable group makes L a solvable Lie algebra.

Lemma 2.7 (Lemma 4.1 of [3])**.** *Suppose that L is an ideal of L. Then L/I is abelian if and only if I contains the derived algebra L'.*

Proof. (Proof quoted from [3] page 28.) The algebra L/I is abelian if and only if for all $x, y \in L$ we have

$$[x + I, y + I] = [x, y] + I = I$$

or, equivalently, for all $x, y \in L$ we have $[x, y] \in I$. Since I is a subspace of L, this holds if and only if the space spanned by the bracktes $[x, y]$ is contained in I; that is, $L' \subseteq I$. □

Lemma 2.8 (Lemma 4.3 of [3])**.** *If L is a Lie algebra with ideals*

$$L = I_0 \supseteq I_1 \supseteq \dots \supseteq I_{m-1} \supseteq I_m = 0$$

such that I_{k-1}/I_k is abelian for $1 \leq i \leq m$, then L is solvable.

Proof. (Proof quoted from [3] pages 29-29.) We shall show that $L^{(k)}$ is contained in I_k for k between 1 and m. Putting $k = m$ will then give $L^{(m)} = 0$.

Since L/I_1 is abelian, we have from Lemma 4.1 that $L' \subseteq I_1$. For the inductive step, we suppose that $L^{(k-1)} \subseteq I_{k-1}$, where $k \geq 2$. The Lie algebra I_{k-1}/I_k is abelian. Therefore by Lemma 4.1, this time applied to the Lie algebra I_{k-1}, we have $[I_{k-1}, I_{k-1}] \subseteq I_k$. But $L^{(k-1)}$ is contained in I_{k-1} by our inductive hypothesis, so we deduce that

$$L^{(k)} = [L^{(k-1)}, L^{(k-1)}] \subseteq [I_{k-1}, I_{k-1}]$$

and hence $L^{(k)} \subseteq I_k$. □

Now we introduce a important particular ideal of every Lie algebra, the radical.

Definition 2.9. *The* ***radical*** *of a Lie algebra L, denoted* $\operatorname{rad} L$*, is the unique maximal solvable ideal of L, that is, if $I \subset L$ is a solvable ideal, then $I \subset \operatorname{rad} L$.*

One must actually prove that $\operatorname{rad} L$ is well-defined; this is shown in Corollary 4.5 of [3].

In addition to the derived series of ideals $L^{(k)}$, there is another important series of ideals of L, called the **central lower series**, denoted by L^k. In the case of the derived series, we had

$$L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$$

The central lower series is defined by the similar recursive formula

$$L^k = [L, L^{k-1}]$$

and one gets a similar sequence of containments:

$$L \supset L^1 \supset L^2 \supset L^3 \supset \dots$$

Definition 2.10. *A Lie algebra L is* ***nilpotent*** *if $L^k = 0$ for some $k \geq 1$.*

As one would expect, a subalgebra of a solvable or nilpotent Lie algebra inherits being solvable or nilpotent, respectively.

Proposition 2.11 (Lemma 4.4(a) of [3])**.** *If L is a solvable Lie algebra, then every subalgebra of L is solvable.*

Proof. Let L be solvable with subalgebra A. Then $L^{(m)} = 0$ for some m. Notice that $A^{(k)} \subseteq L^{(k)}$ for all k, so $A^{(m)} \subseteq L(m) = 0$, thus $A^{(m)} = 0$. □

Proposition 2.12 (Lemma 4.9(a) of [3])**.** *If L is a nilpotent Lie algebra, then every subalgebra of L is nilpotent.*

Proof. Let L be nilpotent with subalgebra A. Then $L^m = 0$ for some m and $A^k \subseteq L^k$ for all k, so $A^k \subseteq L^k = 0$, hence $A^k = 0$. □

2.2 Isomorphism Theorems and Direct Sums

We assume the reader is familiar with definitions of subspaces, cosets, and quotient spaces of vector spaces. Once again, Lie algebras have analogous structures - one can consider cosets of an ideal and impose a bracket structure on them to make the space of cosets a Lie algebra (see [3] section 2.2). Lie algebras have analogous isomorphism theorems to vector spaces and groups.

Theorem 2.13 (Isomorphism theorems)**.**

1. *Let $\phi : L_1 \to L_2$ be a Lie algebra homomorphism. Then $L_1/\ker\phi \cong \operatorname{im}\phi$.*
2. *Let I, J be ideals of a Lie algebra. Then $(I + J)/(J \cong I/(I \cap J)$*
3. *Let I, J be ideals of a Lie algebra with $I \subset J$. Then J/I is an ideal of L/I and $(L/I)/(J/I) \cong L/J$.*

Another construction inherited from vector spaces is that of direct sums.

Definition 2.14. *Let L_1, L_2 be Lie algebras. We define $L_1 \oplus L_2$ to be the vector space direct sum of L_1, L_2 as vector spaces, and give it the bracket*

$$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2])$$

For a proof that this is bracket satisfies the necessary properties, see the solution to Exercise 2.6 in the appendix. As one would expect, the projections

$$\begin{array}{ll} L_1 \oplus L_2 \to L_1 & \qquad L_1 \oplus L_2 \to L_2 \\ (x_1, 0) \mapsto x_1 & \qquad (0, x_2) \mapsto x_2 \end{array}$$

are surjective Lie algebra homomorphisms, so $L_1 \oplus L_2$ contains subalgebras isomorphic to L_1 and L_2. (For more on this, see solution to Exercise 2.7 in the appendix.)

2.3 Simple and Semisimple

Definition 2.15. *A Lie algebra is **simple** if it has no nonzero proper ideals and it is not abelian.*

Note that if $\dim L > 2$ and L has no nonzero proper ideals, then it is simple.

Definition 2.16. *A Lie algebra is **semisimple** if has no nonzero solvable ideals.*

There are many other equivalent characterizations of semisimple Lie algebras, as illustrated by the following proposition.

Proposition 2.17. *Let L be a Lie algebra. The following are equivalent:*

1. *L is semisimple.*
2. $\operatorname{rad} L = 0$.
3. *L has no nonzero abelian ideals.*
4. *L can be written as a direct sum of simple Lie algebras.*

Proof. The equivalence of (1) and (2) is immediate from the definitions. If L is semisimple, then the only solvable ideal is the zero ideal, so $\operatorname{rad} L = 0$. If $\operatorname{rad} L = 0$, then there are no bigger solvable ideals, so L is semisimple.

The equivalence of (2) and (3) is Exercise 4.6. We begin by showing that (2) implies (3). Let L be a semisimple Lie algebra. Let I be a nonzero ideal of L. Suppose I is abelian. Then $[I, I] = 0$, so I is solvable. However, this contradicts the fact that L has no nonzero solvable ideals, so I must not be abelian. Thus L has no nonzero abelian ideals.

Now we show that (3) implies (2). Let L be a Lie algebra with no nonzero abelian ideals. Suppose L has a nonzero solvable ideal I. Then $I^{(k)} = 0$ for some k. Let m be the minimum over such k, so that $I^{(m)} = 0$ but $I^{(m-1)} \neq 0$. Then $[I^{(m-1)}, I^{(m-1)}] = I^{(m)} = 0$ so $I^{(m-1)}$ is a nonzero abelian ideal of L, which is a contradiction. Thus we conclude that L has no solvable ideals. Thus we have shown that (1) is equivalent to (2).

The equivalence of (1) and (4) is proven in Theorem 9.11 of [3]. We defer the proof until later (Theorem 5.5), since it requires machinery that has not yet been defined. □

3 Matrix Lie Algebras

Most important examples of Lie algebras are matrix algebras, and as one eventually discovers in the classification, every finite-dimensional semisimple complex Lie algebra is isomorphic to a matrix algebra, except for five exceptions. For all matrix Lie algebras, the bracket is given by the matrix commutator, $[x, y] = xy - yx$. One can quickly confirm that this bracket always satisfies $[x, x] = 0$ and the Jacobi identity. Before getting into matrix algebras, we need the notation e_{ij}.

Definition 3.1. *The matrix e_{ij} is the matrix with a one in the ijth place and zero elsewhere.*

Lemma 3.2. $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}$

Proof. See appendix, Proposition 6.10. □

Definition 3.3. $\mathrm{gl}(n, F)$ *is the set of $n \times n$ matrices with entries in F.*

It has dimension n^2, since $\{e_{ij} : 1 \le i, j \le n\}$ is a basis. $\mathrm{gl}(n, F)$ is closely related to the following, as indicated by the parallel notation.

Definition 3.4. *Let V be a vector space over a field F. Then* $\mathrm{gl}(V)$ *is the set of linear maps from V to V.*

For any vector space V, $\mathrm{gl}(V)$ is a Lie algebra under the bracket $[x, y] = x \circ y - y \circ x$. (For proof that this bracket satisfies the Jacobi identity, see solution to Exercise 1.3 in the appendix.)

The parallel "gl" notation is justified by the fact that if V is finite-dimensional, then $\mathrm{gl}(V)$ is isomorphic to $\mathrm{gl}(n, F)$. If one fixes a basis of V, then one can represent every map $x \in \mathrm{gl}(V)$ uniquely by a matrix in $\mathrm{gl}(n, F)$, and every matrix in $\mathrm{gl}(n, F)$ gives a linear map in $\mathrm{gl}(V)$. This bijection preserves the bracket since composition of linear maps corresponds to multiplication of matrix representations. Note that in this correspondence, to compute the "action" of a matrix as a linear map on a vector, one multiplies the matrix by the vector, with the matrix as the left multiplicand.

Because of this isomorphism, people often refer to the elements of $\mathrm{gl}(V)$ as matrices, and the same terminology is used to refer to matrices and linear maps. Here's a table summarizing the equivalence. Let V be any finite-dimensional vector space over F (isomorphic to F^n), $v \in V$, and $\lambda \in F$.

	Linear maps	Matrices
ambient space	$\mathrm{gl}(V)$	$\mathrm{gl}(n, F)$
binary operation	composition	multiplication
zero element	zero map, $v \mapsto 0$	matrix of all zeros
identity element	identity map, $v \mapsto v$	identity matrix, $\mathrm{diag}(1, 1, \ldots)$
nilpotent	$\exists n$ such that $x^n(v) = 0, \forall v$	$\exists n$ such that $x^n = 0$
eigenvectors	v such that $x(v) = \lambda v$	v such that $xv = \lambda v$
eigenvalues	λ such that $x(v) = \lambda v$	λ such that $xv = \lambda v$
diagonalizable	$\exists$ basis of V of eigenvectors	$\exists$ basis of V of eigenvectors

3.1 The Adjoint Representation

Definition 3.5. *Let L be a Lie algebra over a field F. A* ***representation*** *of L is a Lie algebra homomorphism $\phi : L \to \mathrm{gl}(V)$ where V is a vector space over F.*

While technically the word "representation" refers to the map ϕ, often this map is described without being named, and instead the vector space V is called a representation of L. Assuming that V is finite-dimensional, $\mathrm{gl}(V)$ is isomorphic to $\mathrm{gl}(n, F)$, so a representation allows one to work with the elements of L as if they were matrices, by considering their images under ϕ. This explains the use of the term "representation;" a representation allows one to represent the elements of L as matrices.

The most important example of a representation is the adjoint representataion.

Definition 3.6. *Let L be a Lie algebra. The* ***adjoint representation*** *of L is the map $\mathrm{ad} : L \to \mathrm{gl}(L)$ where $\mathrm{ad}\, x$ is the map $y \mapsto [x, y]$. That is, $\mathrm{ad}\, x(y) = [x, y]$.*

One can check that ad is a Lie algebra homomorphism (see [3] pages 4-5). If L is finite-dimensional, the adjoint representation always allows one to represent L as a Lie subalgebra of $\mathrm{gl}(n, F)$. You may lose some information about the structure of L, though, since ad is not necessarily one-to-one. (The word **faithful** is often used to refer to a one-to-one representation.) However, even when ad is not faithful, it is always possible to find a faithful representation as a matrix algebra, due to Ado's Theorem.

Theorem 3.7 (Ado)**.** *Let L be a finite-dimensional Lie algebra. There exists a faithful representation of L, $\phi : L \to \mathrm{gl}(V)$ where V is a finite-dimensional vector space.*

Proof. See [9]. For more on why the result holds over arbitrary fields, see [8]. □

This theorem justifies focusing study of the subalgebras of $\mathrm{gl}(n, F)$, since such an approach is in fact the study of all finite-dimensional Lie algebras.

3.2 Matrices with Trace Zero

Definition 3.8. $\mathrm{sl}(n, F)$ *is the set of $n \times n$ matrices with entries in F and trace zero.*

It has dimension $n^2 - 1$, since $\{e_{ij} : i \neq j\} \cup \{e_{ii} - e_{i+1,i+1} : 1 \leq i \leq n - 1\}$ is a basis. To confirm that $\mathrm{sl}(n, F)$ is a Lie algebra, one needs to check that the bracket of two matrices with trace zero also has trace zero. In fact, the bracket of any two matrices has trace zero, as shown in the next proposition.

Proposition 3.9. *Let x, y be matrices. Then $\mathrm{tr}[x, y] = 0$.*

Proof. Let x_{ij} and y_{ij} be the ijth entries of x, y respectively. Then

$$\mathrm{tr}(xy) = \sum_i (xy)_{ii} = \sum_i \sum_j x_{ij} y_{ji} = \sum_j \sum_i y_{ji} x_{ij} = \sum_j (yx)_{jj} = \mathrm{tr}(yx)$$

Hence

$$\mathrm{tr}[x, y] = \mathrm{tr}(xy - yx) = \mathrm{tr}(xy) - \mathrm{tr}(yx) = 0$$

□

This establishes that $\mathrm{sl}(n, F)$ is closed under the bracket. It also establishes that the derived algebra $\mathrm{gl}(n, F)'$ is contained in $\mathrm{sl}(n, F)$. In fact, the derived algebra is not merely a subset of $\mathrm{sl}(n, F)$, but equal to it.

Proposition 3.10 (Exercise 2.10)**.** *The derived algebra of* $\mathrm{gl}(n, F)$ *is* $\mathrm{sl}(n, F)$*.*

Proof. Using the above lemma, we compute:

$$\begin{aligned} [e_{i1}, e_{1j}] &= \delta_{11} e_{ij} = e_{ij} \quad \text{for } i \neq j \\ [e_{i,i+1}, e_{i+1,i}] &= \delta_{i+1,i+1} e_{ii} - \delta_{ii} e_{i+1,i+1} = e_{ii} - e_{i+1,i+1} \quad \text{for } 1 \leq i < n \end{aligned}$$

Thus $e_{ij}, e_{ii} - e_{i+1,i+1} \in \mathrm{gl}(n, F)'$, so $\mathrm{gl}(n, F)'$ contains the basis described above for $\mathrm{sl}(n, F)$. We already know that $\mathrm{gl}(n, F)' \subset \mathrm{sl}(n, F)$, so now that we know $\mathrm{gl}(n, F)'$ is a subspace of equal or greater dimension, $\mathrm{gl}(n, F)'$ must be equal to $\mathrm{sl}(n, F)$. □

One particularly important instance of $\mathrm{sl}(n, F)$ is $\mathrm{sl}(n, \mathbb{C})$, since it appears in the final classification of semisimple complex Lie algebras. It also has the significant property of being simple.

Proposition 3.11 (Exercise 4.7)**.** $\mathrm{sl}(n, \mathbb{C})$ *is a simple Lie algebra for* $n \geq 2$*.*

Proof. See Exercise 4.7 in appendix. □

3.3 Upper Triangular Matrices

We introduce two more matrix Lie algebras, and then examine several of their properties.

Definition 3.12. $\mathrm{b}(n, F)$ *is the set of* $n \times n$ *upper triangular matrices with entries in* F*.*

To check that $\mathrm{b}(n, F)$ is closed under the bracket, note that the product of upper triangular matrices is upper triangular.

Definition 3.13. $\mathrm{n}(n, F)$ *is the set of* $n \times n$ *strictly upper triangular matrices with entries in* F*.*

$\mathrm{n}(n, F)$ is also closed under the bracket, so $\mathrm{b}(n, F)$ and $\mathrm{n}(n, F)$ are subalgebras of $\mathrm{gl}(n, F)$. Note that every strictly upper triangular matrix is nilpotent.

Proposition 3.14. $\mathrm{b}(n, F)' = \mathrm{n}(n, F)$*.*

Proposition 3.15. $\mathrm{n}(n, F)$ *is nilpotent.*

Proposition 3.16. $\mathrm{b}(n, F)$ *is solvable. Additionally, if* $n \geq 2$*, then* $\mathrm{b}(n, F)$ *is not nilpotent.*

For proofs of the above propositions, see Exercises 4.4 and 4.5 in the appendix. At first glance, $\mathrm{b}(n, F)$ being solvable and $\mathrm{n}(n, F)$ being nilpotent seem like very narrow results, but actually there is a sense in which $\mathrm{b}(n, F)$ is a model for many solvable Lie algebras, and $\mathrm{n}(n, F)$ is a model for many nilpotent Lie algebras.

Proposition 3.17 (Exercise 5.4i). *Let L be a Lie subalgebra of* $\mathrm{gl}(V)$. *Suppose there is a basis of V such that every $x \in L$ is represented by a strictly upper triangular matrix. Then L is isomorphic to a subalgebra of* $\mathrm{n}(n, F)$.

Proposition 3.18 (Exercise 5.4ii). *Let L be a Lie subalgebra of* $\mathrm{gl}(V)$. *Suppose there is a basis of V such that every $x \in L$ is represented by an upper triangular matrix. Then L is isomorphic to a subalgebra of* $\mathrm{b}(n, F)$.

See appendix for proofs. Note that one can make the above into "if and only if" statements, since the other direction is straightforward. (If $L \cong \mathrm{b}(n, F)$, then that isomorphism gives a representation of L in which every x is upper triangular, and likewise for $\mathrm{n}(n, F)$.)

These propositions say that $\mathrm{b}(n, F)$ and $\mathrm{n}(n, F)$ are not merely basic instances of solvable and nilpotent Lie algebras; in a loose sense, they encapsulate much of the general structure of solvable and nilpotent Lie algebras. The next two major results extend the above pair of propositions and make this "loose sense" rigorous.

Theorem 3.19 (Engel's Theorem). *Let V be a vector space, and let L be a Lie subalgebra of* $\mathrm{gl}(V)$ *such that every element of L is a nilpotent linear transformation of V. Then there is a basis of V in which every element of L is represented by a strictly upper triangular matrix.*

Proof. See [3] pages 46-48. □

We already knew that if L can be represented as all strictly upper triangular matrices, then L is nilpotent, because of Propositions 3.15 and 3.17. Engel's Theorem says that if there is a faithful representation of L in which the representation of each $x \in L$ is nilpotent, then there is a faithful matrix representation of L in all $x \in L$ are strictly upper triangular. Engel's Theorem is frequently written in a slightly different form:

Theorem 3.20 (Engel's Theorem). *A Lie algebra L is nilotent if and only if for all $x \in L$ the map* $\mathrm{ad}\, x$ *is nilpotent.*

Proof. See [3] pages 48-49. □

Lie's Theorem gives a similar characterization for upper triangular matrices, but requires that the underlying field be $\mathbb{C}$.

Theorem 3.21 (Lie's Theorem). *Let V be a finite-dimensional vector space over $\mathbb{C}$, and let L be a solvable Lie subalgebra of* $\mathrm{gl}(V)$. *Then there is a basis of V in which every element of L is represented by an upper triangular matrix.*

Proof. See [3] pages 49-51. □

3.4 Classical Lie Algebras

We have already defined $\mathrm{sl}(n, F)$; $\mathrm{sl}(n, \mathbb{C})$ is the same, just with the field being $\mathbb{C}$. There are two other families of classical Lie algebras that appear in the classification of semisimple complex Lie algebras.

Definition 3.22. $\mathrm{so}(n, \mathbb{C})$ *is the set of $n \times n$ orthogonal matrices with determinant one. That is,*

$$\mathrm{so}(n, \mathbb{C}) = \{x \in \mathrm{gl}(n, \mathbb{C}) : x^t = x^{-1}, \det x = 1\}$$

In order to define $\mathrm{sp}(2\mathrm{n}, \mathbb{C})$, let Ω be the following $2n \times 2n$ block matrix.

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Definition 3.23. $\mathrm{sp}(2\mathrm{n}, \mathbb{C})$ *is the set of $2n \times 2n$ matrices with entries in $\mathbb{C}$ that satisfy $x^t \Omega x = \Omega$. That is,*

$$\mathrm{sp}(2\mathrm{n}, \mathbb{C}) = \{x \in \mathrm{gl}(2n, \mathbb{C}) : x^t \Omega x = \Omega\}$$

Note that $\mathrm{so}(n, \mathbb{C})$ is a Lie subalgebra of $\mathrm{sl}(n, \mathbb{C})$ and $\mathrm{sp}(2\mathrm{n}, \mathbb{C})$ is a subalgebra of $\mathrm{sl}(2n, \mathbb{C})$ (see [3] page 33).

3.5 Exceptional Lie Algebras

The exceptional Lie algebras are known as e_6, e_6, e_8, f_4, and g_2. There's not much else to say about them here, except that they're finite-dimensional semisimple complex Lie algebras. They'll show up in the big classification of Dynkin diagrams later.

4 Modules

Even though we already have the language of representations to think of any finite-dimensional Lie algebra as a matrix algebra, we introduce another set of notation for thinking of a Lie algebra as a set of linear maps acting on some vector space V, the language of modules.

Definition 4.1. *Let L be a Lie algebra over F. An* ***L-module*** *is a vector space V over F with a map*

$$\begin{aligned} L \times V &\to V \\ (x, v) &\mapsto x \cdot v \end{aligned}$$

with the properties

$$(\lambda x + \mu y) \cdot v = \lambda(x \cdot v) + \mu(y \cdot v) \tag{M 1}$$
$$x \cdot (\lambda v + \mu w) = \lambda(x \cdot v) + \mu(x \cdot w) \tag{M 2}$$
$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) \tag{M 3}$$

There is an exact correspondence between L-modules and representations of L. As mentioned before, one often doesn't care about the name of the map in a representation of L, and viewing it as a module allows one to ignore the map ϕ. Here is a short table summarizing the correspondence between L-modules and representations of L.

	Representations	Modules
main mapping	$\phi : L \to \mathrm{gl}(V)$	$L \times V \to V, (x, v) \mapsto x \cdot v$
image of one element	$\phi(x)(v)$	$x \cdot v$
linearity in first entry	ϕ is linear	(M1)
linearity in second entry	$\phi(x)$ is linear $\forall x$	(M2)
preserves bracket	ϕ is homomorphism	(M3)
submodule/subrepresentation	$x \in L, w \in W \to \phi(x)(w) \in W$	$x \in L, w \in W \to x \cdot w \in W$
homomorphism	$\theta(\phi(x)v) = \psi(x)\theta(v)$	$\theta(x \cdot v) = x \cdot \theta(v)$

As shown on pages 55-56 of [3], one can make a representation into an L-module and one can make an L-module into a representation.

Definition 4.2. *Let V be an L-module. A* ***submodule*** *of V is a subspace W such that for $x \in L, w \in W$, we have $x \cdot w \in W$.*

When first encountering submodules, one often sees the confusing interchangability of "submodule of V" and "submodule of L." Even though the phrase "submodule of L" implies a subspace of L, it refers to the same thing, that is, subspaces of V.

Definition 4.3. *Let V be an L-module. V is* ***irreducible*** *if it is nonzero and has no nonzero proper submodules.*

There are, of course, many modules that are not irreducible. One would hope that we could always decompose a module as a direct sum of irreducible submodules. While this is not true for all Lie algebras, it is true for semisimple complex Lie algebras; this result is Weyl's Theorem.

Theorem 4.4 (Weyl's Theorem)**.** *Let L be a complex semisimple Lie algebra. Every finite dimensional L-module can be written as a direct sum of irreducible submodules.*

Proof. See appendix B of [3]. $\square$

As with any algebraic structure, modules/representations have their own notion of homomorphism.

Definition 4.5. *Let L be a Lie algebra and let V, W be L-modules. An* ***L-module homomorphism*** *is a linear map $\theta : V \to W$ with*

$$\theta(x \cdot v) = x \cdot \theta(v)$$

for all $v \in V, x \in L$.

Definition 4.6. *Let L be a Lie algebra and let $\phi : L \to \mathrm{gl}(V), \psi : L \to \mathrm{gl}(W)$ be representations. A representation homomorphism is a linear map $\theta : V \to W$ with*

$$\theta(\phi(x)v) = \psi(x)\theta(v)$$

for all $v \in V, x \in L$.

Of course, the definitions for homomorphism are equivalent, but we have separated them into two different definitions since the notation looks quite different. As usual, an isomorphism (denoted $\cong$) is a bijective homomorphism.

Lemma 4.7 (Schur's Lemma). *Let L be a complex Lie algebra and let S be a finite-dimensional irreducible L-module. A map $\theta : S \to S$ is an L-module homomorphism if and only if θ is a scalar multiple of the identity transformation, that is, if $\theta = \lambda 1_S$ for some $\lambda \in \mathbb{C}$.*

Proof. See [3] page 62. $\square$

4.1 Classification of $\mathrm{sl}(2, \mathbb{C})$ modules

Somewhat surprisingly, the irreducible $\mathrm{sl}(2, \mathbb{C})$ modules are completely determined by dimension. First we need to define the common basis e, f, h for $\mathrm{sl}(2, \mathbb{C})$.

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Definition 4.8. *V_d is the vector space of homogenous polynomials of degree d in two variables with complex coefficients.*

The natural basis for V_d is $\{x^d, x^{d-1}y, \dots y^{d-1}x, y^d\}$. It has dimension $d+1$. V_d can be viewed as an $\mathrm{sl}(2, \mathbb{C})$ module by defining

$$e \cdot x = x\frac{\partial}{\partial y} \qquad f \cdot x = y\frac{\partial}{\partial x} \qquad h \cdot x = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

For details on how this is in fact a module, and that it is irreducible, see pages 67-70 of [3].

Theorem 4.9 (Theorem 8.5 of [3]). *Let V be a finite-dimensional irreducible $\mathrm{sl}(2, \mathbb{C})$ module. Then V is isomorphic to V_d for some d.*

Proof. See [3] pages 71-73. $\square$

5 Classification of Semisimple Complex Lie Algebras

We now look at the structures needed for the classification theorem of Killing and Cartan.

5.1 The Killing Form

As mentioned in the historical notes, the Killing form is somewhat improperly named, as it was actually first introduced by Cartan. But someone (not Cartan) decided to call this map the Killing form, and the name stuck.

Definition 5.1. *Let L be a complex Lie algebra. The* ***Killing form*** *on L is the symmetric bilinear form $\kappa : L \times L \to \mathbb{C}$ defined by*

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y)$$

Note that κ is symmetric because $\operatorname{tr}(ab) = \operatorname{tr}(ba)$ for matrices a, b. The values of the Killing form allow one to deduce a lot about the structure of L, as seen in the following two propositions.

Theorem 5.2 (Cartan's First Criterion)**.** *A complex Lie algebra L is solvable if and only if $\kappa(x, y) = 0$ for all $x \in L, y \in L'$.*

Proof. See [3] pages 80-81. □

Theorem 5.3 (Cartan's Second Criterion)**.** *A complex Lie algebra L is semisimple if and only if the Killing form on L is non-degenerate.*

Proof. See pages 82-83 of [3] □

We now have the required machinery to prove the equivalence of (1) and (4) asserted in Proposition 2.17, that a Lie algebra is semisimple if and only if it can be written as a direct sum of simple ideals. First we prove one more lemma.

Lemma 5.4 (Lemma 9.10 of [3])**.** *If I is a non-trivial proper ideal in a complex semisimple Lie algebra L, then $L = I \oplus I^{\perp}$. The ideal I is a semisimple Lie algebra in its own right.*

Proof. (This proof quoted from [3], page 83.) As usual, let κ denote the Killing form on I. The restriction of κ to $I \cap I^{\perp}$ is identically 0, so by Cartan's First Criterion, $I \cap I^{\perp} = 0$. It now follows by dimension counting that $L = I \oplus I^{\perp}$.

We shall show that I is semisimple using Cartan's Second Criterion. Suppose that I has a non-zero solvable ideal. By the "only if" direction of Cartan's Second Criterion, the Killing form on I is degenerate. We have seen that the Killing form on I is given by restricting the Killing form on L, so there exists $a \in I$ such that $\kappa(a, x) = 0$ for all $x \in I$. But as $a \in I, \kappa(a, y) = 0$ for all $y \in I^{\perp}$ as well. Since $L = I \oplus I^{\perp}$, this shows that κ is degenerate, a contradiction. □

Theorem 5.5 (Theorem 9.11 of [3])**.** *Let L be a complex Lie algebra. Then L is semisimple if and only if there are simple ideasl $L_1, \dots L_r$ of L such that $L = L_1 \oplus L_2 \oplus \dots \oplus L_r$.*

Proof. (This proof quoted for [3], pages 83-84.) We begin with the "only if" direction, working by induction on $\dim L$. Let I be an ideal in L of the smallest possible non-zero dimension. If $I = L$, we are done. Otherwise I is a proper simple ideal of L. (It cannot be abelian as by hypothesis L has non non-zero abelian ideals.) By the preceding lemma, $L = I \oplus I^{\perp}$, where, as an ideal of L, $I^{\perp}$ is a semisimple Lie algebra of smaller dimension than L. So, by induction, $I^{\perp}$ is a direct sum of simple ideals,

$$I^{\perp} = L_2 \oplus \dots \oplus L_r$$

Each L_i is also an ideal of L, as $[I, L_i] \subseteq I \cap I^{\perp} = 0$, so putting $L_1 = I$ we get the required decomposition.

Now for the "if" direction. Suppose that $L = L_1 \oplus \dots \oplus_r$, where the L_i are simple ideals. Let $I = \operatorname{rad} L$; our aim is to show that $I = 0$. For each ideal $L_i, [I, L_i \subseteq I \cap L_i$ is a solvable ideal of L_i. But the L_i are simple, so

$$[I, L] \subseteq [I, L_1] \oplus \dots \oplus [I, L_r] = 0$$

This shows that I is contained in $Z(L)$. But by Exercise 2.6(ii) (see appendix)

$$Z(L) = Z(L_1) \oplus \ldots \oplus Z(L_r).$$

We know that $Z(L_1) = \ldots = Z(L_r) = 0$ as the L_i are simple ideals, so $Z(L) = 0$ and $I = 0$. $\square$

5.2 Root space decomposition

In order to classify the semisimple complex Lie algebras, we will see that they can all be decomposed as a direct sum of a Cartan subalgebra with a bunch of root spaces. The big picture is that the structure of the root spaces determines the Lie algebra up to isomorphism.

Definition 5.6. *Let L be a Lie algebra. A* ***Cartan subalgebra*** *is a maximal Lie subalgebra with two properties: H is abelian, and for $h \in H$, $\operatorname{ad} h$ is diagonalizable.*

On page 95, [3] shows that every semisimple complex Lie algebra has a Cartan subalgebra. However, it is not unique.

Definition 5.7. *Let L be a semisimple complex Lie algebra with Cartan subalgebra H. A* ***root space*** *corresponding to the* ***root*** *α is the space*

$$L_\alpha = \{x \in L : [h, x] = \alpha(h)x, \forall h \in H\}$$

In general, $\alpha \in H^*$ is only called a root when the corresponding root space L_α is nonzero.

It does not seem obvious at all from the definition, but all of the root spaces L_α are always one-dimensional ([3] Proposition 10.9).

The Killing form κ is a symmetric bilinear form on L, but eventually we plan to view the roots $\alpha \in H^*$ as living in an inner product space, so that we can think of them as root systems (to be defined shortly). We use κ to define this inner product, though the definition is rather involved.

For $h \in H$, define $\theta_h : H \to \mathbb{C}$ by $\theta_h(k) = \kappa(h, k)$. The map $H \to H^*$ given by $h \to \theta_h$ is an isomorphism. (For more details than you probably wanted, see page 99 of [3]). Now we can define a bilinear form on H^*,

$$(,) : H^* \times H^* \to \mathbb{C} \qquad (\theta_h, \theta_k) = \kappa(h, k)$$

Proposition 5.8 (Proposition 10.15 of [3])**.** *Let β be a basis for H^* consisting of roots of L. The above form is a real-valued inner product on the real span of β.*

As a useful convention, we define another binary operator on the same space, in terms of this inner product.

$$\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}$$

Definition 5.9. *Let L be a semisimple complex Lie algebra with Cartan subalgebra H. The* ***root space decomposition*** *of L is the direct sum expression*

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

where Φ is the set of roots, that is,

$$\Phi = \{\alpha \in H^* : \alpha \neq 0, L_\alpha \neq 0\}$$

5.3 Root systems

Before we can define a root system, we need to define the reflection s_α. For $\alpha \neq 0$ in an inner product space E, s_α is the reflection through the hyperplane perpendicular to α. The following formula is useful for computations.s

$$s_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\beta, \beta)}\beta = \beta - \langle \alpha, \beta \rangle \beta$$

Definition 5.10. *A **root system** is a subset R of a real inner-product space E satisfying*

1. *R is finite, R spans E, and $0 \notin R$.*
2. *For $\alpha \in R$, the only multiples of α in R are $\pm\alpha$.*
3. *For $\alpha \in R$, the reflection s_α permutes R.*
4. *For $\alpha, \beta \in R$, $\langle \beta, \alpha \rangle \in \mathbb{Z}$.*

For our purposes, the most salient fact is that the roots of a semisimple complex Lie algebra are a root system, given in the following proposition.

Proposition 5.11. *Let L be a semisimple complex Lie algebra with Cartan subalgebra H and roots Φ. Let E be the real span of Φ. Then Φ is a root system in E.*

Proof. See Example 11.2 on page 110 of [3]. □

We noted previously that a given Lie algebra may have various Cartan subalgebras, the root space decomposition is not unique. Since we plan to show that a root system of a Lie algebra determines the Lie algebra up to isomorphism, we need to establish that different root space decompositions give rise to the same root systems. In order to talk about the "sameness" of root systems, we need a notion of isomorphism.

Definition 5.12. *Let R, R' be root systems in the real inner-product spaces E, E' respectively. A **root system isomorphism** is a vector space isomorphism $\phi : E \to E'$ such that $\phi(R) = R'$ and $\langle \alpha, \beta \rangle = \langle \phi(\alpha), \phi(\beta) \rangle$ for $\alpha, \beta \in R$. If there is an isomorphism bewteen Φ_1 and Φ_2, we write $\Phi_1 \cong \Phi_2$.*

This allows us to state the following proposition, which resolves the "sameness" question raised above.

Theorem 5.13 (Theorem 12.6 of [3])**.** *Let L be a complex semisimple Lie algebra. Let L be a semisimple complex Lie algebra with Cartan subalgebras H_1, H_2 and corresponding root systems Φ_1, Φ_2. Then $\Phi_1 \cong \Phi_2$.*

Proof. See appendix C of [3]. □

As with modules, root systems are unwieldy in general, so we try to decompose them into more basic versions that have restrictive properties.

Definition 5.14. *A root system R is **irreducible** if R cannot be expressed as a disjoint union of two nonempty subset R_1, R_2 such that $(\alpha, \beta) = 0$ for all $\alpha \in R_1, \beta \in R_2$.*

This property is important because of the following lemma.

Lemma 5.15 (Lemma 11.8 of [3])**.** *Every root system may be written as a disjoint union of irreducible root systems.*

The following proposition makes an important link between irreducible root systems and simple complex Lie algebras.

Proposition 5.16. *Let L be a complex semisimple Lie algebra with Cartan subalgebra H and root system Φ. Φ is irreducible if and only if L is simple.*

Proof. The "if" direction is Proposition 12.4 of [3] (found on pages 128-129), and the "only if" direction is Proposition 14.2 (found on page 154). □

Since every root system can be written as a disjoint union of irreducible root systems, and the irreducible root systems correspond to simple Lie algebras, and semisimple Lie algebras can be written as direct sums of simple Lie algebras, we see why perhaps semisimple Lie algebras are determined by their root systems. To make that determination rigorous, we need Dynkin diagrams.

5.4 Cartan Matrix and Dynkin Diagrams

We first consider an important substructure of a root system.

Definition 5.17. *A **base** for a root system R is a subset $B \subset R$ such that B is a basis for E and every $\beta \in R$ can be written in the form*

$$\beta = \sum_{\alpha \in B} k_\alpha \alpha$$

where $k_\alpha \in \mathbb{Z}$ and all the nonzero k_α have the same sign.

This allows us to sort the elements of R into two buckets: ones where the nonzero k_α are positive, and ones where the k_α are negative. These two subsets are called R^+ and R^- respectively.

Definition 5.18. *Let $B = (\alpha_1, \alpha_2, \dots \alpha_n)$ be an ordered base for the root system R. The **Cartan matrix** is the $n \times n$ matrix where the ijth entry is $\langle \alpha_i, \alpha_j \rangle$.*

This definition clearly depends upon the choice of base and ordering. Surprisingly, the Cartan matrix of a root system does not depend on the choice of base (see Theorem 11.16 and appendix D of [3]).

Definition 5.19. *Let B be an ordered base of a root system R. The **Dynkin diagram** of R is the graph with one vertex for each root $\alpha \in B$, and between each pair of vertices α, β, there are $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ lines. Additionally, whenever $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle > 1$, there is an arrow pointing towards whichever of α, β is longer. Alternately, one can replace arrows and instead color longer vertices a different color.*

Note that $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ is always an integer between 0 and 4 inclusive (see Lemma 11.4 of [3]), so it makes sense to talk about that as a number of lines. Note that a Dynkin diagram contains exactly the same information as a Cartan matrix.

Proposition 5.20. *Let Φ be a root system. The Φ is irreducible if and only if its Dynkin diagram is connected.*

Proof. We will prove the contrapositive of both directions. First suppose that a Dynkin diagram corresponding to R is disconnected. Then there are two disjoint sets of vertices R_1, R_2 which have no lines between them. Thus for $\alpha \in R_1, \beta \in R_2$,

$$0 = \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)} \frac{2(\alpha, \beta)}{(\alpha, \alpha)} = \frac{4(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)}$$

hence $(\alpha, \beta) = 0$. Thus R is reducible.

Now suppose that R is reducible. Then there exist disjoint subsets R_1, R_2 with $(\alpha, \beta) = 0$ for $\alpha \in R_1, \beta \in R_2$. Then R_1, R_2 correspond to disjoint subgraphs of the Dynkin diagram. Hence the Dynkin diagram is disconnected. □

Next, we connect Dynkin diagrams to root systems.

Proposition 5.21 (Proposition 11.21 of [3])**.** *Let R, R' be root systems in the real vector spaces E, E' respectively. If the Dynkin diagrams of R and R' are the same, then the root systems are isomorphic.*

Proof. See pages 122-123 of [3]. □

Finally, we get the theorem that relates the Cartan matrix to the isomorphism class of a semisimple complex Lie algebra, though the statement sort of goes the opposite direction. Don't worry, though, the useful part (for classifying) comes as a neatly stated corollary. First, we need to define generators of a Lie algebra.

Definition 5.22. *Let L be a Lie algebra. A set $\{x_1, \ldots x_n\}$ **generates** L if every $x \in L$ can be written as a linear combination of $x_1, \ldots x_n$ and brackets of $x_1, \ldots x_n$.*

Theorem 5.23 (Serre's Theorem)**.** *Let C be the Cartan matrix of a root system. Let L be the complex Lie algebra which is generated by elements e_i, f_i, h_i for $1 \leq i \leq l$ satisfying*

$$[h_i, h_j] = 0 \qquad [h_i, e_j] = c_{ji} e_j \qquad [h_i, f_j] = -c_{ji} f_j \qquad [e_i, f_i] = h_i$$

for all i, j, and

$$[e_i, f_j] = 0 \qquad (\operatorname{ad} e_i)^{1-c_{ji}}(e_j) = 0 \qquad (\operatorname{ad} f_i)^{1-cji}(f_j) = 0$$

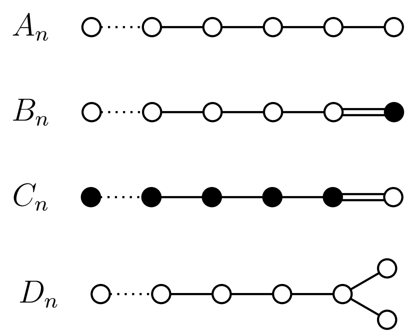
for $i \neq j$. Then L is finite-dimensional and semisimple wth Cartan subalgebra H spanned by $\{h_1, \ldots h_l\}$, and its root system has Cartan matrix C.

Theorem 5.24 (Corollary of Serre's Theorem)**.** *Let L_1, L_2 be complex semisimple Lie algebras with Cartan matrix C. Then L_1 is isomorphic to L_2.*

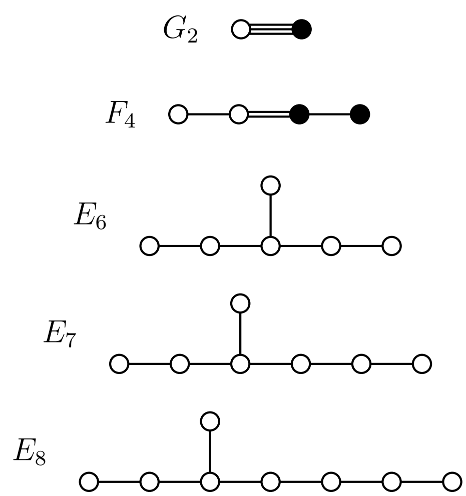
So finally we have demonstrated what has been hinted at, that the Cartan matrix of a complex semisimple Lie algebra allows one to categorize the Lie algebra into an isomorphism class. Since we've also related the Cartan matrix to a root system and a Dynkin diagram, all that remains is to classify the Dynkin diagrams.

5.5 Final Classification

Theorem 5.25 (Theorem 13.1 of [3])**.** *Given an irreducible root system R, the unlabelled Dynkin diagram associated to R is either a member of the four families*



where each of the diagrams above has n vertices, or one of the five exceptional diagrams



Proof. See chapter 13 of [3]. □

This gives rise to the classification theorem for simple complex Lie algebras.

Theorem 5.26. *Every finite-dimensional simple complex Lie algebra is isomorphic to one of*

$$\mathrm{sl}(n, \mathbb{C}) \qquad \mathrm{so}(n, \mathbb{C}) \qquad \mathrm{sp}(2\mathrm{n}, \mathbb{C})$$

for some n, or to one of the five exceptional Lie algebras, e_6, e_7, e_8, f_4, g_2.

Proof. See chapter 12 of [3]. □

As implied by the labelling scheme, the exceptional Lie algebras correspond to the capitalized Dynkin diagram names. For the others, this table gives the correspondence.

Graph	Lie algebra
A_n	$\mathrm{sl}(n+1,\mathbb{C})$
B_n	$\mathrm{so}(2n+1,\mathbb{C})$
D_n	$\mathrm{so}(2n,\mathbb{C})$
C_n	$\mathrm{sp}(2n,\mathbb{C})$

Keep in mind that the above classification of Dynkin diagrams is a classification of connected Dynkin diagrams, but of course any Dynkin diagram is a union of connected components. Each connected component corresponds to an irreducible root system, which corresponds to a simple complex Lie algebra. A disconnected Dynkin diagram D corresponds to a semisimple complex Lie algebra, where each direct summand is a simple Lie algebra corresponding to the connected components of D.

Lie algebra	Root system	Dynkin diagram
direct sum	disjoint union	union of disjoint, connected subgraphs
simple	irreducible	connected

Appendix: Solutions to Exercises

6 Chapter 1 Exercises

Proposition 6.1 (Exercise 1.1i). *Let L be a Lie algebra, and let $v \in L$. Then* $[v, 0] = [0, v] = 0$.

Proof. By bilinearity of the bracket,

$$[v, 0] = [v, v - v] = [v, v] - [v, v] = 0$$
$$[0, v] = [v - v, v] = [v, v] - [v, v] = 0$$

□

Note that in the following proposition the symbol "0" is used for both the additive identity in the field F and the additive identity vector in L. When added to a vector, "0" refers to a vector; when multiplied by a bracket or vector "0" refers to the identity for F.

Lemma 6.2 (Lemma for Exercise 1.1ii). *Let L be a Lie algebra over F. Let $x, y \in L$ and $a \in F$. Then $a[x, y] = [ax, y] = [x, ay]$.*

Proof.

$$a[x, y] = a[x, y] + 0[0, y] = [ax + 0, y] = [ax, y]$$
$$a[x, y] = 0[x, 0] + a[x, y] = [x, ay + 0] = [ax, y]$$

□

Proposition 6.3 (Exercise 1.1ii). *Let L be a Lie algebra with $x, y \in L$ such that $[x, y] \neq 0$. Then x and y are linearly independent over F.*

Proof. Let $x, y \in L$ such that $[x, y] \neq 0$. Suppose x, y are linearly dependent over F. Then there exist $a, b \in F$ with $a \neq 0, b \neq 0$ such that $ax + by = 0$, or equivalently $ax = -by$.

Let $v = [x, y]$. Then $av = a[x, y] = [ax, y]$ and $-bav = -b[ax, y] = [ax, -by]$. Since $a \neq 0, b \neq 0$ and $v \neq 0$, thus $-abv \neq 0$ so $[ax, -by] \neq 0$. However, we showed that $ax = -by$, so by the property of the Lie bracket, $[ax, -by] = 0$. Thus we reject our hypothesis that x, y are linearly dependent and conclude that they are linearly independent. □

Lemma 6.4 (Lemma for Exercise 1.2). *For any vectors $u, v, w \in \mathbb{R}^3$,* $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$.

Proof.

$$\begin{aligned}
u \times (v \times w) &= u \times (v^2w^3 - v^3w^2, v^3w^1 - v^1w^3, v^1w^2 - v^2w^1) \\
&= \big(u^2(v^1w^2 - v^2w^1) - u^3(v^3w^1 - v^1w^3), \\
&\quad u^3(v^2w^3 - v^3w^2) - u^1(v^1w^2 - v^2w^1), \\
&\quad u^1(v^3w^1 - v^1w^3) - u^2(v^2w^3 - v^3w^2)\big) \\
&= (v^1u^2w^2 - v^2u^2w^1 - v^3u^3w^1 + v^1u^3w^3, \\
&\quad v^2u^3w^3 - v^3u^3w^2 - u^1v^1w^2 + v^2u^1w^1, \\
&\quad v^3u^1w^1 - v^1v^1w^3 - u^2v^2w^3 +^3 u^2w^2) \\
&= (v^1u^2w^2 - v^2u^2w^1 - v^3u^3w^1 + v^1u^3w^3 + v^1u^1w^1 - v^1u^1w^1, \\
&\quad v^2u^3w^3 - v^3u^3w^2 - u^1v^1w^2 + v^2u^1w^1 + v^2u^2w^2 - v^2u^2w^2, \\
&\quad v^3u^1w^1 - v^1v^1w^3 - u^2v^2w^3 +^3 u^2w^2 + v^3u^3w^3 - v^3u^3w^3) \\
&= \big(v^1(u^1w^1 + u^2w^2 + u^3w^3) - w^1(u^1v^1 + u^2v^2 + u^3v^3), \\
&\quad v^2(u^1w^1 + u^2w^2 + u^3w^3) - w^2(u^1v^1 + u^2v^2 + u^3v^3), \\
&\quad v^3(u^1w^1 + u^2w^2 + u^3w^3) - w^3(u^1v^1 + u^2v^2 + u^3v^3)\big) \\
&= \big(v^1(u \cdot w), v_2(u \cdot w), v^3(u \cdot w)\big) - \big(w^1(u \cdot v), w^2(u \cdot w), w^3(u \cdot w)\big) \\
&= (u \cdot w)v - (u \cdot v)w
\end{aligned}$$

□

Proposition 6.5 (Exercise 1.2). *The Jacobi identity holds for the cross product of vectors in* $\mathbb{R}^3$*.*

Proof. Using the above proposition,

$$\begin{aligned}
[x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= (x \cdot z)y - (x \cdot y)z \\
&\quad + (y \cdot x)z - (y \cdot z)x \\
&\quad + (z \cdot y)x - (z \cdot x)y \\
&= (x \cdot z)y - (z \cdot x)y \\
&\quad + (y \cdot x)z - (x \cdot y)z \\
&\quad + (z \cdot y)x - (y \cdot z)x \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned}$$

□

Proposition 6.6 (Exercise 1.3). *Let* V *be a finite-dimensional vector space over* F *and let* $\mathrm{gl}(V)$ *be the set of all linear maps from* V *to* V*. We define a Lie bracket on this space by*

$$[x, y] := x \circ y - y \circ x$$

where $\circ$ *denotes map composition. We claim that the Jacobi identity holds for this bracket operator.*

Proof.

$$\begin{aligned}
[x,[y,z]]+[y,[z,x]]+[z,[x,y]] &= (x\circ y\circ z - z\circ y\circ x)\\
&\quad + (y\circ z\circ x - x\circ z\circ y)\\
&\quad + (z\circ x\circ y - y\circ x\circ z)\\
&= (x\circ y\circ z - x\circ z\circ y)\\
&\quad + (y\circ z\circ x - y\circ x\circ z)\\
&\quad + (z\circ x\circ y - z\circ y\circ x)\\
&= x\circ [y,z] + y\circ [z,x] + z\circ [x,y]\\
[x,[y,z]]+[y,[z,x]]+[z,[x,y]] &= (x\circ y\circ z - y\circ x\circ z)\\
&\quad + (z\circ x\circ y - x\circ z\circ y)\\
&\quad + (y\circ z\circ x - z\circ y\circ x)\\
&= [x,y]\circ z + [z,x]\circ y + [y,z]\circ x
\end{aligned}$$

Thus we reach

$$x\circ [y,z] + y\circ [z,x] + z\circ [x,y] = [x,y]\circ z + [z,x]\circ y + [y,z]\circ x$$

Now we can subtract to have one side equal zero,

$$\begin{aligned}
0 &= x\circ [y,z] - [y,z]\circ x + y\circ [z,x] - [z,x]\circ y + z\circ [x,y] - [x,y]\circ z\\
&= [x,[y,z]]+[y,[z,x]]+[z,[x,y]]
\end{aligned}$$

which is precisely the Jacobi identity. □

Note that for $n\times n$ matrices A, B with entries A_{ij}, B_{ij}, the entries of the matrix product AB are

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} \tag{6.1}$$

Definition 6.7. *The* ***Kronecker delta function*** *is the function*

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{6.2}$$

Sometimes the alternate notation δ^i_j is used (equivalent to δ_{ij}).

Definition 6.8. *In* $\mathrm{gl}(n,F)$*, e_{ij} is the matrix with a 1 in position ij and zero everwhere else. Using the delta function, we can say that the ab-th entry of e_{ij} is $\delta_{ai}\delta_{bj}$.*

Proposition 6.9 (Lemma for Page 3)**.** *In* $\mathrm{gl}(n,F)$*, the product of the matrices e_{ij} and e_{kl} is given by the formula:*

$$(e_{ij}e_{kl})_{ab} = \delta_{ai}\delta_{bl}\delta_{jk} \tag{6.3}$$

Proof. By the general rule for square matrix products,

$$(e_{ij}e_{kl})_{ab} = \sum_{x=1}^{n} (e_{ij})_{ax}(e_{kl})_{xb}$$

Note that the term to be summed over is equal to 1 when $a = i, j = x, k = x$, and $l = b$. Since x ranges over $1, 2, \dots n$ and $1 \leq j, k \leq n$, this summand is 1 when $a = i, b = l$, and $j = k = x$. In each summand, x takes a different value, so no two terms can both have $x = j = k$, but if $j = k$, then in one term we will have $x = j = k$. Thus

$$\begin{aligned}
(e_{ij}e_{kl})_{ab} &= \sum_{x=1}^{n} (e_{ij})_{ax}(e_{kl})_{xb} \\
&= \begin{cases} 1 & a = i, b = l, \text{and} j = k \\ 0 & \text{otherwise} \end{cases} \\
&= \delta_{ai}\delta_{bl}\delta_{jk}
\end{aligned}$$

□

Proposition 6.10 (Page 3)**.** *In* gl(n, F)*, the bracket of two basis matrices is given by*

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj} \tag{6.4}$$

Proof.

$$\begin{aligned}
[e_{ij}, e_{kl}]_{ab} &= (e_{ij}e_{kl} - e_{kl}e_{ij})_{ab} \\
&= \delta_{ai}\delta_{bl}\delta_{jk} - \delta_{ak}\delta_{bj}\delta_{li} \\
(\delta_{jk}e_{il} - \delta_{il}e_{kj})_{ab} &= \delta_{jk}(e_{il})_{ab} - \delta_{il}(e_{kj})_{ab} \\
&= \delta_{jk}\delta_{ai}\delta_{lb} - \delta_{il}\delta_{ka}\delta_{bj}
\end{aligned}$$

Since $\delta_{ij} = \delta_{ji}$ for any i, j, by commuting these expressions we see that they are equal. Since the two matrices have equal entries for all $1 \leq a, b \leq n$, they are equal matrices. □

Proposition 6.11 (Exercise 1.4)**.** *Let* b(n, F) *be the subset of upper triangular matrices in* gl(n, F)*. Then* b(n, F) *is a Lie algebra with the same bracket as in* gl(n, F)*.*

Proof. We need to show that $b(n, F)$ is closed under the bracket. Let $x, y \in b(n, F)$. Since the product or sum of two upper triangular matrices is itself upper triangular, $[x, y] = xy - yx$ is upper triangular, so it is also an element of $b(n, F)$. □

Proposition 6.12 (Exercise 1.4)**.** *Let* n(n, F) *be the strictly upper triangular matrices in* gl(n, F)*. Then* n(n, F) *is a Lie algebra with the same bracket.*

Proof. Same argument as previous proposition. □

Proposition 6.13 (Exercise 1.5)**.** *Let $L = sl(2, F)$. If char$(F) \neq 2$ then*

$$Z(\mathrm{sl}(2, F)) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

and if char$(F) = 2$ then

$$Z(\mathrm{sl}(2, F)) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Proof. Let $L = \mathrm{sl}(2, F)$. The following matrices form a basis for L.

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

And the bracket products of these are

$$[e, f] = g \qquad [f, g] = 2f \qquad [g, e] = 2e$$

by straightforward computation. Let $y \in L$ and let $x \in Z(L)$. Then $xy = yx$. We write x and y as linear combinations of the basis matrices e, f, g

$$x = x_{11}g + x_{12}e + x_{21}f = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix}$$
$$y = y_{11}g + y_{12}e + y_{21}f = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & -y_{11} \end{pmatrix}$$

Then we use the linearity of the bracket to compute

$$[x, y] = \begin{vmatrix} x_{12} & x_{21} \\ y_{12} & y_{21} \end{vmatrix} g + 2 \begin{vmatrix} x_{11} & x_{12} \\ y_1 & y_{12} \end{vmatrix} e - 2 \begin{vmatrix} x_{11} & x_{21} \\ y_{11} & y_{21} \end{vmatrix} f$$

Since g, e, f are linearly independent over F, from this we know that

$$\begin{vmatrix} x_{12} & x_{21} \\ y_{12} & y_{21} \end{vmatrix} = x_{12}y_{21} - x_{21}y_{12} = 0$$

for all $y_1, y_2 \in F$. This is true only if $x_{12} = x_{21} = 0$, regardless of the characteristic of F. Similarly, the coefficients of e and f must be zero, which is true only when $2 = 0$ or when $x_{11} = 0$. If $\mathrm{char}(F) = 2$, then we see that g is actually the identity matrix, which clearly commutes with everything, but in this case there are no other possible values for x_{11}, so the center of L is just the zero matrix and the identity matrix.

However, if $\mathrm{char}(F) \neq 2$, then we must have $x_{11} = 0$, so the only matrix in the center of L is the zero matrix. □

Proposition 6.14 (Exercise 1.6)**.** *Let L_1, L_2 be Lie algebras and let $\phi : L_1 \to L_2$ be a homomorphism. Then* $\ker \phi$ *is an ideal of L_1.*

Proof. We need to show that for $x \in L_1, y \in \ker \phi = \{v \in L_1 : \phi(v) = 0\}$, we have $[x, y] \in \ker \phi$. Let $x \in L_1, y \in \ker \phi$. Then $\phi([x, y]) = [\phi(x), \phi(y)] = [\phi(x), 0] = 0$. □

Proposition 6.15 (Exercise 1.6). *Let $\phi : L_1 \to L_2$ be a Lie algebra homomorphism. Then $\operatorname{im}\phi$ is a subalgebra of L_2.*

Proof. We need to show that for $x, y \in \operatorname{im}\phi$, we have $[x, y] \in \operatorname{im}\phi$. Let $x, y \in \operatorname{im}\phi$. Then there exist $x', y' \in L_1$ such that $\phi(x') = x, \phi(y') = y$. Then $[x', y'] \in L_1$, so $\phi([x', y']) \in \operatorname{im}\phi$. Since $\phi([x', y']) = [\phi(x'), \phi(y')] = [x, y]$, we see that $[x, y] \in \operatorname{im}\phi$. □

Proposition 6.16 (Exercise 1.7). *Let L be a Lie algebra such that for all $a, b \in L$, we get $[a, b] \in Z(L)$. Then the Lie bracket is associative.*

Proof. Let $x, y, z \in L$ Then $[x, y] \in Z(L)$, so $[[x, y], z] = 0$. We also know that $[y, z] \in Z(L)$, so we get that $[x, [y, z]] = [-[y, z], x] = 0$ so $[x, [y, z]] = [[x, y], z] = 0$. □

Proposition 6.17 (Exercise 1.7). *Let L be a Lie algebra such that the bracket is associative. Then for $x, y \in L$, $[x, y] \in Z(L)$.*

Proof. Let $x, y, z \in L$. We need to show that $[[x, y], z] = 0$. Using anti-communitivity, linearity, and associativity we get

$$[z, [x, y]] = -[[x, y], z] = -[-[y, x], z] = [[y, x], z] = [y, [x, z]] = [y, -[z, x]] = -[y, [z, x]]$$

Then using the Jacobi identity and substituting $-[y, [z, x]]$ for $[z, [x, y]]$

$$\begin{aligned}
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0 \\
[x, [y, z]] + [y, [z, x]] - [y, [z, x]] &= 0 \\
[x, [y, z]] &= 0 \\
[[x, y], z] &= 0
\end{aligned}$$

□

Proposition 6.18 (Exercise 1.8i). *Let D, E be derivations of an algebra A. Then $[D, E] = D \circ E - E \circ D$ is a derivation (of A).*

Proof. We need to show that $[D, E](xy) = x[D, E](y) + [D, E](x)y$. First we compute $D \circ E(xy)$ and $E \circ D(xy)$.

$$\begin{aligned}
D \circ E(xy) &= D(xE(y) + E(x)y) \\
&= D(xE(y)) + D(E(x)y) \\
&= xD \circ E(y) + D(x)E(y) + D \circ E(x)y + E(x)D(y) \\
E \circ D(xy) &= xE \circ D(y) + E(x)D(y) + D(x)E(y) + E \circ D(x)y
\end{aligned}$$

Now that we've done that we can easily compute $[D, E](xy)$.

$$\begin{aligned}
[D, E](xy) &= (D \circ E - E \circ D)(xy) \\
&= D \circ E(xy) - E \circ D(xy) \\
&= xD \circ E(y) + D \circ E(x)y - xE \circ D(y) - E \circ D(x)y \\
&= x(D \circ E(y) - E \circ D(y)) + (D \circ E(x) - E \circ D(x))y \\
&= x[D, E](y) - [D, E](x)y
\end{aligned}$$

□

Proposition 6.19 (Exercise 1.9)**.** *Let L_1, L_2 be Lie algebras such that there exist bases β_1 for L_1 and β_2 for L_2 such that the structure constants of L_1 with respect to β_1 are equal to the structure constants of L_2 with respect to β_2. Then $L_1 \cong L_2$.*

Proof. Let $\beta_1 = \{x_1, x_2 \dots x_n\}$ and $\beta_2 = \{y_1, y_2 \dots y_n\}$ be bases for L_1, L_2 as described. Then since the structure constants are the same,

$$[x_i, x_j] = \sum_{k=1}^{n} a_{ij}^k x_k$$
$$[y_i, y_j] = \sum_{k=1}^{n} a_{ij}^k y_k$$

We define a linear map $\phi : L_1 \to L_2$ by $\phi(x_i) = y_i$. Since ϕ maps β_1 to β_2, ϕ is a bijection. Also,

$$\phi([x_i, x_j]) = \phi\left(\sum_{k=1}^{n} a_{ij}^k x_k\right) = \sum_{k=1}^{n} a_{ij}^k \phi(x_k) = \sum_{k=1}^{n} a_{ij}^k y_k = [y_i, y_j] = [\phi(x_i), \phi(x_j)]$$

so ϕ is an isomorphism. □

Proposition 6.20 (Exercise 1.9)**.** *Let L_1, L_2 be isomorphic Lie algebras. Then there exist bases β_1 for L_1 and β_2 for L_2 such that the structure constants for L_1 with respect to β_1 are equal to the strcture constants for L_2 with respect to β_2.*

Proof. Let $\beta_1 = \{x_1, x_2 \dots x_n\}$ be any basis for L_1 and let $\phi : L_1 \to L_2$ be an isomorphism. Let $\beta_2 = \{\phi(x_1), \phi(x_2), \dots \phi(x_n)\}$. Since ϕ is a linear bijection, β_2 is a basis for L_2. Let a_{ij}^k be the structure constants of L_1 with respect to β_1. Then

$$[\phi(x_i), \phi(x_j)] = \phi([x_i, x_j]) = \phi\left(\sum_{k=1}^{n} a_{ij}^k x_k\right) = \sum_{k=1}^{n} a_{ij}^k \phi(x_k)$$

so by definition a_{ij}^k are the structure constants of L_2 with respect to β_2. □

Proposition 6.21 (Exercise 1.10)**.** *Let L be a Lie algebra with basis $\{x_1, x_2 \dots x_n\}$ and structure constants with respect to this basis a_{ij}^k. Then*

$$a_{jk}^m a_{im}^t + a_{ki}^m a_{jm}^t + a_{ij}^m a_{km}^t = 0$$

for $1 \le i, j, k, m, t \le n$.

Proof. From the Jacobi identity

$$
\begin{aligned}
0 &= [x_i,[x_j,x_k]] + [x_j,[x_k,x_i]] + [x_i,[x_j,x_k]] \\
&= [x_i, \sum_{m=1}^{n} a_{jn}^m x_m] + [x_j, \sum_{m=1}^{n} a_{ki}^m x_m] + [x_k, \sum_{m=1}^{n} a_{ij}^m x_m] \\
&= \sum_{m=1}^{n} a_{jk}^m [x_i, x_m] + \sum_{m=1}^{n} a_{ki}^m [x_j, x_m] + \sum_{m=1}^{n} a_{ij}^m [x_k, x_m] \\
&= \sum_{m=1}^{n} a_{jk}^m \left(\sum_{t=1}^{n} a_{im}^t x_t \right) + \sum_{m=1}^{n} a_{ki}^m \left(\sum_{t=1}^{n} a_{jm}^t x_t \right) + \sum_{m=1}^{n} a_{ij}^m \left(\sum_{t=1}^{n} a_{km}^t x_t \right) \\
&= \sum_{m=1}^{n} \sum_{t=1}^{n} a_{jk}^m a_{im}^t x_t + \sum_{m=1}^{n} \sum_{t=1}^{n} a_{ki}^m a_{jm}^t x_t + \sum_{m=1}^{n} \sum_{t=1}^{n} a_{ij}^m a_{km}^t x_t \\
&= \sum_{m=1}^{n} \sum_{t=1}^{n} \left(a_{jk}^m a_{im}^t + a_{ki}^m a_{jm}^t + a_{ij}^m a_{km}^t \right) x_t
\end{aligned}
$$

since x_t for $t = 1, 2, \ldots n$ are linearly independent, this implies that the coefficient $a_{jk}^m a_{im}^t + a_{ki}^m a_{jm}^t + a_{ij}^m a_{km}^t$ is equal to zero for all values of i, j, k, m, t. □

Lemma 6.22 (Lemma for Exercies 1.11)**.** *Let V, W be n-dimensional vector spaces over a field F. Then $V \cong W$. (vector space isomorphism)*

Proof. Let $\{v_i\}_{i=1}^n, \{w_i\}_{i=1}^n$ be bases for V and W respectively. Let $\phi : V \to W$ be a linear map defined on v_i by $\phi(v_i) = (w_i)$ for $i = 1, 2, \ldots n$. Then for a general vector in V given by $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$, we compute

$$
\begin{aligned}
\phi(a_1 v_1 + a_2 v_2 + \ldots + a_n v_n) &= a_1 \phi(v_1) + a_2 \phi(v_2) + \ldots + a_n \phi(v_n) \\
&= a_1 w_1 + a_2 w_2 + \ldots + a_n w_n
\end{aligned}
$$

Since every element of W can be written uniquely as a linear combination of $w_1, \ldots w_n$, from this we get that ϕ is one-to-one and onto. Thus ϕ is an isomorphism. □

Proposition 6.23 (Exercise 1.11)**.** *Let L_1, L_2 be n-dimensional abelian Lie algebras over F. Then $L_1 \cong L_2$. (Lie algebra isomorphism)*

Proof. As shown above, L_1 and L_2 are isomorphic as vector space via the map ϕ. We can see that ϕ is also a Lie algebra isomorphism for abelian Lie algebras since

$$\phi([x, y]) = \phi(0) = 0 = [\phi(x), \phi(y)]$$

□

Proposition 6.24 (Exercise 1.8ii)**.** *Let $A = C^\infty \mathbb{R}$ be the vector space of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. Let $D : A \to A$ be the usual derivative $D(f) = f'$. Then $D \circ D$ is not a derivation for the algebra A.*

Proof.

$$\begin{aligned} D \circ D(fg) &= D(f'g + fg') \\ &= D(f'g) + D(fg') \\ &= f''g + f'g' + fg'' + fg'g \\ &= D \circ D(f)g + fD \circ D(g) + 2f'g' \end{aligned}$$

If $D \circ D$ were a derivation, it would satisfy the above equation only when $2f'g' = 0$ for all $f, g \in A$. But this is not true, since $f(x) = x, g(x) = x$ gives $2f'g' = 2$. □

Lemma 6.25 (Lemma for Exercise 1.11)**.** *Let V, W be finite-dimensional isomorphic F-vector spaces. Then* $\dim V = \dim W$.

Proof. Let $\{v_i\}_{i=1}^n$ be a basis for V, and let $\phi : V \to W$ be an isomorphism. We claim that $\{\phi(v_i)\}_{i=1}^n$ is a basis for W. To do this, we just need to show that $\{\phi(v_i)\}_{i=1}^n$ is linearly independent.

Let $a_1, a_2 \dots a_n \in F$ such that

$$a_1\phi(v_1) + a_2\phi(v_2) + \dots a_n\phi(v_n) = 0$$

Then by linear of ϕ,

$$\phi(a_1v_1 + a_2v_2 + \dots + a_nv_n) = 0$$

Since ϕ is one-to-one, $\ker \phi = \{0\}$, so the above equation implies that

$$a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$$

Since $\{v_i\}$ is a basis, it is linearly independent, so the above implies that $a_i = 0$ for $i = 1, 2, \dots n$. Thus $\{\phi(v_i)\}$ is linearly independent and thus a basis of size n for W, so $\dim W = n = \dim V$. □

Proposition 6.26 (Exercise 1.11)**.** *Let L_1, L_2 be finite-dimensional, isomorphic abelian Lie algebras. Then* $\dim L_1 = \dim L_2$.

Proof. By the above lemma, L_1, L_2 have equal dimension as vector spaces. □

Solution 0.1 (Exercise 1.12)**.** The structure constants of $sl(2, F)$ with respect to the basis

$$x_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad x_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are

$$\begin{array}{ccc} a_{12}^1 = 0 & a_{13}^1 = -2 & a_{23}^1 = 0 \\ a_{12}^2 = 0 & a_{13}^2 = 0 & a_{23}^2 = 2 \\ a_{12}^3 = 0 & a_{13}^3 = 0 & a_{23}^3 = 0 \end{array}$$

Proposition 6.27 (Exercise 1.13)**.** $\mathrm{sl}(2, \mathbb{C})$ *has no non-trivial proper ideals.*

Proof. We will show that any non-zero ideal of $\mathrm{sl}(2,\mathbb{C})$ is the whole space. Suppose $I \subseteq \mathrm{sl}(2,\mathbb{C})$ with $I \neq \{0\}$. Then there exists $x \in I$ with $x \neq 0$. Let $x = ae + bf + ch$, where e, f, h are the matrices x_1, x_2, x_3 from Exerise 1.12. Since $x \neq 0$, at least one of $a, b, c \neq 0$.

Since I is an ideal, $[h, x] \in I$, and we compute

$$[h,x] = a[h,e] + b[h,f] = 2ae - 2bf$$

Furthermore, since I is an ideal, $[e,[h,x]] \in I$ and $[f,[h,x]] \in I$, and we compute

$$\begin{aligned} [e,[h,x]] &= -2b[e,f] = -2bh \\ [f,[h,x]] &= -2ah \end{aligned}$$

If $a \neq 0$ or $b \neq 0$, then $h \in I$. If $h \in I$, then $e, f \in I$ because $[f,h] = 2f$ and $[h,e] = 2e$.

Suppose $a = 0$ and $b = 0$. Then $x = ch$ so $h \in I$, so then $e, f \in I$. Thus if a, b are both zero or at least one is nonzero, then $h \in I$, and if $h \in I$, then $e, f \in I$. Thus for any values of a, b, c, $e, f, h \in I$, so I contains a basis for $\mathrm{sl}(2,\mathbb{C})$, so $I = \mathrm{sl}(2,\mathbb{C})$. □

Proposition 6.28 (Exercise 1.14i)**.** *Let L be the 3-dimensional Lie algebra over $\mathbb{C}$ with basis $\{x, y, z\}$ where the bracket is defined by*

$$[x,y] = z, \;\; [y,z] = x, \;\; [z,x] = y$$

Then L is isomorphic to the Lie subalgebra of $\mathrm{gl}(3,\mathbb{C})$ consisting of antisymmetric matrices.

Proof. Let A be the subalgebra of $\mathrm{gl}(3,\mathbb{C})$ of antisymmetric matrices. Take the basis a, b, c for A where

$$a = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

One can crunch numbers and compute that

$$[a,b] = c, \;\; [b,c] = a, \;\; [c,a] = b$$

We define a map $\phi : L \to A$ by $\phi(x) = a, \phi(y) = b, \phi(z) = c$. Since a, b, c and x, y, z are bases, ϕ is a bijection, and by the above in conjunction with bilinearity, ϕ preserves bracket products. Thus ϕ is an isomorphism. □

Proposition 6.29 (Exercise 1.14ii)**.** *Let L be the 3-dimensional Lie algebra over $\mathbb{C}$ with basis $\{x, y, z\}$ where the bracket is defined by*

$$[x,y] = z, \;\; [y,z] = x, \;\; [z,x] = y$$

Then $L \cong \mathrm{sl}(2,\mathbb{C})$. (Lie algebra isomorphism)

Proof. Let $(\hat{x}, \hat{y}, \hat{z})$ be the matrices

$$\hat{x} = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \hat{y} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \hat{z} = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

We claim that these matrices are linearly independent over $\mathbb{C}$. Let $u, v, w \in \mathbb{C}$ such that $u\hat{x} + v\hat{y} + w\hat{z} = 0$. Then clearly $u = 0$, since the upper left entry of $\hat{y}$ and $\hat{z}$ are zero. From the other matrix positions we get the equations

$$\begin{aligned} v/2 + w/2 &= 0 \\ -v/2 + w/2 &= 0 \end{aligned}$$

Adding these equations gives $w = 0$, and then plugging in $w = 0$ into one of them gives $v = 0$. Thus, $(\hat{x}, \hat{y}, \hat{z})$ are linearly independent over $\mathbb{C}$, and since $\mathrm{sl}(2, \mathbb{C})$ is a three dimensional vector space, they must also span it, and thus be a basis for $\mathrm{sl}(2, \mathbb{C})$.

Straightforward commputation of the bracket products in $\mathrm{sl}(2, \mathbb{C})$ of these matrices gives

$$\begin{aligned}
[\hat{x}, \hat{y}] &= \frac{1}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = \hat{z} \\
[\hat{y}, \hat{z}] &= \frac{1}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \frac{1}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = \hat{x} \\
[\hat{z}, \hat{x}] &= \frac{1}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \hat{y}
\end{aligned}$$

Finally, we define a map $\phi : \mathrm{sl}(2, \mathbb{C}) \to L$ by $\phi(\hat{x}) = x, \phi(\hat{y}) = y, \phi(\hat{z}) = z$. Since both (x, y, z) and $(\hat{x}, \hat{y}, \hat{z})$ are bases, ϕ is a bijection, and by our computations we've seen that ϕ preserves the bracket operator. Thus, ϕ is an isomorphism of Lie algebras. □

Proposition 6.30 (Exercise 1.15i)**.** *Let $S \in \mathrm{gl}(n, F)$. Define*

$$\mathrm{gl}_S(n, F) := \{x \in \mathrm{gl}(n, F) : x^T S = -Sx\}$$

where x^T denotes the transpose of x. Then $\mathrm{gl}_S(n, F)$ is a subalgebra of $\mathrm{gl}_(n, F)$.

Proof. We need to show that $\mathrm{gl}_S(n, F)$ is closed under vector addition, scalar multiplication, and the bracket product. First we show closure under addition. Let $x, y \in \mathrm{gl}_S(n, F)$. Then

$$(x + y)^T S = (x^T + y^T)S = x^T S + y^T S = -Sx - Sy = -S(x + y)$$

so $x + y \in \mathrm{gl}_S(n, F)$. Now let $x \in \mathrm{gl}_S(n, F)$ and $a \in F$. Then

$$(ax)^T S = a(x^T)S = a(x^T S) = a(-Sx) = -S(ax)$$

so $ax \in \mathrm{gl}_S(n, F)$. Finally, for $x, y \in \mathrm{gl}_S(n, F)$,

$$\begin{aligned}
[x, y]^T S = (xy - yx)^T S &= (xy)^T S - (yx)^T S = y^T x^T S - x^T y^T S \\
&= -y^T Sx + x^T Sy = Syx - Sxy = S(yx - xy) = S[y, x] \\
&= -S[x, y]
\end{aligned}$$

Thus $[x, y] \in \mathrm{gl}_S(n, F)$. □

Proposition 6.31 (Exercise 1.15ii). *Let S be*

$$S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then

$$\mathrm{gl}_S(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R} \right\}$$

Proof. Let $x = \mathrm{gl}_S(2, \mathbb{R})$ where

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} -c & d \\ 0 & 0 \end{pmatrix}$$

Thus $a = d, c = 0, d = 0$. Thus

$$\mathrm{gl}_S(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R} \right\}$$

□

Proposition 6.32 (Exercise 1.15iii). *There is no matrix $S \in \mathrm{gl}(2, \mathbb{R})$ such that*

$$\mathrm{gl}_S(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

Proof. Let D be

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

and suppose there exists $S \in \mathrm{gl}(2, \mathbb{R})$ such that $\mathrm{gl}_S(2, \mathbb{R}) = D$. let $x \in D$, where

$$x = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$$

for some fixed $x_1, x_2 \in \mathbb{R}$. Since $x \in \mathrm{gl}_S(2, \mathbb{R})$, $x^T S = -Sx$ so $xS = -Sx$.

$$\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} = -\begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$$
$$\begin{pmatrix} x_1 s_1 & x_1 s_2 \\ x_2 s_3 & x_2 s_4 \end{pmatrix} = \begin{pmatrix} -x_1 s_1 & -x_2 s_2 \\ -x_1 s_3 & -x_2 s_4 \end{pmatrix}$$

so we have four equations, one for each position. By the first, we must have $x_1 = 0$ or $s_1 = 0$, by the second $x_1 = -x_2$ or $s_2 = 0$, by the third, $x_1 = -x_2$ or $s_3 = 0$, and by the fourth, $x_2 = 0$ or $s_4 = 0$.

Clearly, there are matrices in D which do not have $a = 0, b = 0$, or $a = -b$. Yet this matrix equation must still hold for such matrices, so we conclude that $S = 0$, the zero matrix. However, $\mathrm{gl}_0(2, \mathbb{R}) = \mathrm{gl}(2, \mathbb{R}) \neq D$. Thus no such S exists. □

Proposition 6.33 (Exercies 1.15iv)**.** $\mathrm{gl}_{I_3}(3, \mathbb{R}) \cong \mathbb{R}^3_\wedge$ *(Lie algebra isomorphism)*

Proof. Let $i = (1, 0, 0), j = (0, 1, 0), k = (0, 0, 1)$ be the standard basis for $\mathbb{R}^3_\wedge$, and note that $i \times j = k, j \times k = i$, and $k \times i = j$. It turns out that $\mathrm{gl}_{I_3}(3, \mathbb{R})$ is the subalgebra of anti-symmetric matrices, which as shown previously has basis

$$a = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

where $[a, b] = c, [b, c] = a$, and $[c, a] = b$. We define $\phi : \mathbb{R}^3_\wedge \to \mathrm{gl}_{I_3}(3, \mathbb{R})$ by $\phi(i) = a, \phi(b) = j, \phi(k) = c$. Since i, j, k and a, b, c are bases and ϕ preserves brackets, ϕ is an isomorphism. □

Proposition 6.34 (Exercise 1.16)**.** *If F is a field of characteristic 2, then there exist algebras over F which satisfy anticommuntivity $[x, y] = -[y, z]$ and the Jacobi identity but not $[x, x] = 0$.*

Proof. Let A be the algebra on $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ with basis $x_1 = (1, 0)$,$x_2 = (0, 1)$. We define a bilinear map $[,] : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \times \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ by the table

	0	x_1	x_2	$x_1 + x_2$
0	0	0	0	0
x_1	0	$x_1 + x_2$	$x_1 + x_2$	0
x_2	0	$x_1 + x_2$	$x_1 + x_2$	0
$x_1 + x_2$	0	0	0	0

This bracket is symmetric as visible from the table, and since everything has order 2, it is thus antisymmetric. The Jacobi identity is seen to be true because every bracket product is either 0 or $x_1 + x_2$, and every bracket involving 0 or $x_1 + x_2$ is zero. Thus $[x, [y, z]] = 0$ for $x, y, z \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$. However, $[x_1, x_1] \neq 0$, so this is not a Lie bracket. □

Proposition 6.35 (Exercise 1.17)**.** *Let V be a vector space over $\mathbb{C}$ with basis $\beta = \{v_i\}_{i=1}^n$. Let $x : V \to V$ be a diagonalisable linear map with eigenvalues λ_i, that is, $x(v_i) = \lambda_i v_i$. Then $\operatorname{ad} x : \mathrm{gl}(V) \to \mathrm{gl}(V)$ is diagonlisable with eignvalues $\lambda_i - \lambda_j$ for $1 \leq i, j \leq n$.*

Proof. Let $y_{ij} : V \to V$ be the map with matrix $e_{ij} \in \mathrm{gl}(n, \mathbb{C})$. We will show that $\operatorname{ad} x(y_{ij}) = (\lambda_i - \lambda_j)y_{ij}$, so y_{ij} for $1 \leq i, j \leq n$ are eigenvectors of $\operatorname{ad} x$ with eigenvalues $\lambda_i - \lambda_j$. From this we also will get that $\{y_{ij}\}_{1 \leq i,j \leq n}$ is a basis for $\mathrm{gl}(V)$.

First we will show that $e_{ij}[x] = \lambda_j e_{ij}$ and $[x]e_{ij} = \lambda_i e_{ij}$. The matrix of x is

$$[x] = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & \vdots & & \vdots & 0 \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

so the abth entry of $[x]$ is $[x]_{ab} = \delta_{ab}\lambda_a = \delta_{ab}\lambda_b$. The abth entry of e_{ij} is $(e_{ij})_{ab} = \delta_{ai}\delta_{jb}$. Using the formula $(AB)_{ab} = \sum_{k=1}^n A_{ak}B_{kb}$ for the product of two matrices,

$$(e_{ij}[x])_{ab} = \sum_{k=1}^n (e_{ij})_{ak}[x]_{kb} = \sum_{k=1}^n \delta_{ai}\delta_{jk}\delta_{kb}\lambda_b = \delta_{ai}\lambda_b \sum_{k=1}^n \delta_{jk}\delta_{kb}$$

Since $1 \le j, b \le n$, in the sum $\sum_{k=1}^n \delta_{jk}\delta_{kb}$, all terms will be zero except when $j = k = b$, and if $j = b$ then there will be a nonzero term since k ranges from 1 to n. Thus this sum term is equal to δ_{jb}. Note also that because of the factor δ_{jb}, the only time this product is nonzero is when $j = b$, so we can replace λ_b with λ_j. Thus

$$(e_{ij}[x])_{ab} = \delta_{ai}\lambda_b\delta_{jb} = \lambda_j(e_{ij})_{ab}$$

Thus

$$e_{ij}[x] = \lambda_j e_{ij}$$

Using the product formula for matrices again,

$$([x]e_{ij})_{ab} = \sum_{k=1}^n [x]_{ak}(e_{ij})_{kb} = \sum_{k=1}^n \delta_{ak}\lambda_a\delta_{ik}\delta_{jb} = \lambda_a\delta_{jb}\sum_{k=1}^n \delta_{ak}\delta_{ik} = \lambda_a\delta_{ai}\delta_{jb} = \lambda_i(e_{ij})_{ab}$$

so

$$[x]e_{ij} = \lambda_i e_{ij}$$

Now we are able to compute $\operatorname{ad} x(y_{ij})$.

$$\operatorname{ad} x(y_{ij}) = [x, y_{ij}] = x \circ y_{ij} - y_{ij} \circ x$$

We know that $x \circ y_{ij}$ has matrix $[x]e_{ij} = \lambda_i e_{ij}$ and that $y_{ij} \circ x$ has matrix $e_{ij}[x] = \lambda_j e_{ij}$. Then by the previous lemmas about the linearity of the matrix of a transformation, $x \circ y_{ij} - y_{ij} \circ x$ has matrix $\lambda_i e_{ij} - \lambda_j e_{ij} = (\lambda_i - \lambda_j)e_{ij}$. Then the map corresponding to this matrix must be $(\lambda_i - \lambda_j)y_{ij}$. Thus $\operatorname{ad} x(y_{ij}) = (\lambda_i - \lambda_j)y_{ij}$ as desired. $\square$

Proposition 6.36 (Exercise 1.18)**.** *Let L be a Lie algebra over F. Let*

$$\mathrm{IDer}(L) = \{\operatorname{ad} x : x \in L\}$$

Then $\mathrm{IDer}(L)$ *is an ideal of* $\mathrm{Der}(L)$*.*

Proof. First we show that $\mathrm{IDer}(L)$ is closed under addition. Let $x, y \in L$, $\mathrm{ad}\, x, \mathrm{ad}\, y \in \mathrm{IDer}(L)$.

$$(\mathrm{ad}\, x + \mathrm{ad}\, y)(z) = \mathrm{ad}\, x(z) + \mathrm{ad}\, y(z) = [x, z] + [y, z] = [x + y, z] = \mathrm{ad}(x + y)(z)$$

Thus $\mathrm{ad}\, x + \mathrm{ad}\, y$ is in $\mathrm{IDer}(L)$. Now we show that $\mathrm{IDer}(L)$ is closed under scalar multiplication. Let $a \in F$.

$$a\, \mathrm{ad}\, x(z) = a[x, z] = [ax, z] = \mathrm{ad}(ax)(z)$$

Thus $a\, \mathrm{ad}\, x$ is in $\mathrm{IDer}(L)$. Now we show that IDer satisfies the ideal property. Let $D \in \mathrm{Der}(L), \mathrm{ad}\, x \in \mathrm{IDer}(L)$. Then

$$\begin{aligned}[D, \mathrm{ad}\, x](z) &= D \circ adx(z) - \mathrm{ad}\, x \circ D(z) \\ &= D([x, z]) - [x, D(z)] \\ &= [x, D(z)] + [D(x), z] - [x, D(z)] \\ &= [D(x), z] \\ &= ad(D(x))(z)\end{aligned}$$

Thus $[D, \mathrm{ad}\, x] = \mathrm{ad}\, D(x)$, so $[D, \mathrm{ad}\, x] \in \mathrm{IDer}(L)$. $\square$

Proposition 6.37 (Exercise 1.19)**.** *Let A be an algebra and let $\delta : A \to A$ be a derivation. Then*

$$\delta^n(xy) = \sum_{r=0}^{n} \binom{n}{r} \delta^r(x)\delta^{n-r}(y)$$

for all $x, y \in A$.

Proof. Clearly this is true for $n = 1$ by the definition of a derivation. We proceed by induction on n. Suppose the statement is true for $n - 1$, that is,

$$\delta^{n-1}(xy) = \sum_{r=0}^{n-1} \binom{n-1}{r} \delta^r(x)\delta^{n-1-r}(y)$$

for all $x, y \in A$. Then

$$\begin{aligned}\delta^n(xy) &= \delta(\delta^{n-1}(xy)) \\ &= \sum_{r=0}^{n-1} \binom{n-1}{r} \delta^{r+1}(x)\delta^{n-(r+1)}(y) + \delta^r(x)\delta^{n-r}(y) \\ &= \sum_{r=0}^{n-1} \binom{n-1}{r} \delta^{r+1}(x)\delta^{n-(r+1)}(y) + \sum_{r=0}^{n-1} \binom{n-1}{r} \delta^r(x)\delta^{n-r}(y)\end{aligned}$$

Now set $s = r + 1$. Then this is equal to

$$\begin{aligned}
&= \sum_{s=1}^{n} \binom{n-1}{s-1} \delta^s(x)\delta^{n-s}(y) + \sum_{r=0}^{n-1} \binom{n-1}{r} \delta^r(x)\delta^{n-r}(y) \\
&= \sum_{s=0}^{n} \binom{n-1}{s-1} \delta^s(x)\delta^{n-s}(y) + \sum_{r=0}^{n} \binom{n-1}{r} \delta^r(x)\delta^{n-r}(y) \\
&= \sum_{r=0}^{n} \binom{n-1}{r-1} \delta^r(x)\delta^{n-r}(y) + \binom{n-1}{r} \delta^r(x)\delta^{n-r}(y) \\
&= \sum_{r=0}^{n} \left(\binom{n-1}{r-1} + \binom{n-1}{r} \right) \delta^r(x)\delta^{n-r}(y)
\end{aligned}$$

It is a basic identity that

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$$

thus we finally have

$$\delta^n(xy) = \sum_{r=0}^{n} \delta^r(x)\delta^{n-r}(y)$$

Since the statement is true for $n = 1$ and if true for $n - 1$ it must be true for n, thus the statement is true for $n \in \mathbb{N}$ by the principle of induction. $\square$

7 Chapter 2 Exercises

Proposition 7.1 (Exercise 2.1)**.** *Let I, J be ideals of a Lie algebra L. Then*

$$I + J := \{x + y : x \in I, y \in J\}$$

is an ideal of L.

Proof. We need to show that $I + J$ is a vector subspace of L and that for $a \in L, b \in I + J$, we have $[a, b] \in I + J$.

Let $v, w \in I + J$. Then $v = v_i + v_j$ and $w = w_i + w_j$ where $v_i, w_i \in I$ and $v_j, w_j \in J$. Then $v + w = (v_i + v_j) + (w_i + w_j) = (v_i + w_i) + (v_j + w_j)$. Since I, J are vector subspaces, $v_i + w_i \in I$ and $v_j + w_j \in J$. Thus $v + w \in I + J$.

Let $\lambda \in F$. Then $\lambda v = \lambda(v_i + v_j) = \lambda v_i + \lambda v_j$. Since I, J are vector subspaces, $\lambda v_i \in I$ and $\lambda v_j \in J$. Thus $\lambda v \in I + J$.

Let $a \in L, b \in I + J$. Then $b = b_i + b_j$ so

$$[a, b] = [a, b_i + b_j] = [a, b_i] + [a, b_j]$$

Since I, J are ideals of L, $[a, b_i] \in I$ and $[a, b_j] \in J$. Thus $[a, b] \in I + J$. □

Definition 7.2. *Let I, J be ideals of a Lie algebra L. Then we define*

$$[I, J] := Span\{[x, y]x \in I, y \in J\}$$

Proposition 7.3 (Exercise 2.2)**.** $sl(2, \mathbb{C})' = sl(2, \mathbb{C})$

Proof. Take the basis for $sl(2, \mathbb{C})$ given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We compute the bracket products

$$[e, f] = g \qquad [f, g] = 2f \qquad [g, e] = 2e$$

Since g, e, f are linearly independent over $\mathbb{C}$, so are $g, 2f, 2e$. Thus $sl(2, \mathbb{C})'$ has dimension at least 3. Since $sl(2, \mathbb{C})'$ is a span of vectors in $sl(2, \mathbb{C})$, it is a subspace. A subspace of equal dimension must be the whole space. Thus $sl(2, \mathbb{C})' = sl(2, \mathbb{C})$. □

Proposition 7.4 (Exercise 2.4)**.** *Let L be a Lie algebra. Then $L/Z(L)$ is isomorphic to a subalgebra of* $\mathrm{gl}(L)$.

Proof. Consider the map $\mathrm{ad} : L \to \mathrm{gl}(L)$ where $\mathrm{ad}(x) = \mathrm{ad}_x$ is the map $\mathrm{ad}_x : L \to L$ given by $\mathrm{ad}_x(y) = [x, y]$. As shown on pages 4-5 of Erdmann and Wildon, ad is linear and bracket-preserving, with $\ker(\mathrm{ad}) = Z(L)$. As shown in Exercise 1.6, $\mathrm{im}(\mathrm{ad})$ is a subalgebra of $\mathrm{gl}(L)$. By the First Isomorphism Theorem,

$$\begin{aligned} L/\ker(\mathrm{ad}) &\cong \mathrm{im}(\mathrm{ad}) \\ L/Z(L) &\cong \mathrm{im}(\mathrm{ad}) \end{aligned}$$

Thus $L/Z(L)$ is isomorphic to a subalgebra of $\mathrm{gl}(L)$. □

Proposition 7.5 (Exercise 2.3i). *Let L be a Lie algebra over F, and let I be an ideal of L. We define a bracket on L/I by*

$$[w+I, z+I] = [w,z] + I$$

We claim that this bracket is bilinear.

Proof. Let $\lambda_1, \lambda_2 \in F$ and $v_1, v_2, w \in L$. Then

$$\begin{aligned}
[\lambda_1(v_1+I) + \lambda_2(v_2+I), w+I] &= [(\lambda_1 v_1 + I) + (\lambda_2 v_2 + I), w+I] \\
&= [(\lambda_1 v_1 + \lambda_2 v_2) + I, w+I] \\
&= [\lambda_1 v_1 + \lambda_2 v_2, w] + I \\
&= ([\lambda_1 v_1, w] + [\lambda_2 v_2, w]) + I \\
&= ([\lambda_1 v_1 w] + I) + ([\lambda_2 v_2, w] + I) \\
&= (\lambda_1 [v_1, w] + I) + (\lambda_2 [v_2, w] + I)
\end{aligned}$$

thus the bracket is linear in the first component.

$$\begin{aligned}
[w+I, \lambda_1(v_1+I) + \lambda_2(v_2+I)] &= [w+I, (\lambda_1 v_1 + \lambda_2 v_2) + I] \\
&= [w, \lambda_1 v_1 + \lambda_2 v_2] + I \\
&= ([w, \lambda_1 v_1] + [w, \lambda_2 v_2]) + I \\
&= ([w, \lambda_1 v_1] + I) + ([w, \lambda_2 v_2] + I) \\
&= (\lambda_1 [w, v_1] + I) + (\lambda_2 [w, v_2] + I)
\end{aligned}$$

thus the bracket is linear in the second component. This shows that the bracket is bilinear. □

Proposition 7.6 (Exercise 2.3i). *Let L be a Lie algebra over F and let I be an ideal of L. Then the bracket on L/I satisfies*

$$[x,x] = 0$$

for $x \in L/I$.

Proof. Let $v \in L$, so $v + I \in L/I$. Then

$$[v+I, v+I] = [v,v] + I = 0 + I$$

where $0 + I$ is the identity for L/I. □

Proposition 7.7 (Exercies 2.3i). *Let L be a Lie algebra over F, and let I be an ideal of L. The the bracket on L/I satisfies the Jacobi identity.*

Proof. Let $u+I, v+I, w+I \in L/I$. Then

$$[u+I,[v+I,w+I]] + [v+I,[w+I,u+I]] + [w+I,[u+I,v+I]]$$

$$\begin{aligned}
&= [u+I, [v,w]+I] + [v+I, [w,u]+I] + [w+I, [u,v]+I] \\
&= \big([u,[v,w]] + I\big) + \big([v,[w,u]] + I\big) + \big([w,[u,v]] + I\big) \\
&= \big([u,[v,w]] + [v,[w,u]] + [w,[u,v]]\big) + I \\
&= 0 + I
\end{aligned}$$

where $0 + I$ is the additive identity of L/I. □

Proposition 7.8 (Exercise 2.3ii)**.** *Let I be an ideal of Lie algebra L over F. Define*

$$\pi : L \to L/I$$

by $\pi(z) = z + I$. Then π is a Lie algebra homomorphism.

Proof. First we show that π is a linear map. Let $a \in F$ and $u, v \in L$

$$\pi(au + v) = (au + v) + I = (au + I) + (v + I) = a(u + I) + (v + I) = a\pi(u) + \pi(v)$$

so π is linear. Now we show that π preserves the bracket.

$$\pi([u, v]) = [u, v] + I = [u + I, v + I] = [\pi(u), \pi(v)]$$

Thus π is a Lie algebra homomorphism. □

Proposition 7.9 (Exercise 2.5)**.** *Let L be a Lie algebra, and let $v \in L'$. Then $\operatorname{tr}\operatorname{ad} v = 0$.*

Proof. Let $x, y, z \in L$. Then

$$\begin{aligned}
\operatorname{ad}[x, y](z) &= [[x, y], z] \\
&= -[z, [x, y]] && \text{by anticommutativity} \\
&= [x, [y, z]] + [y, [z, x]] && \text{by Jacobi identity} \\
&= [x, [y, z]] - [y, [x, z]] \\
&= \operatorname{ad} x \circ \operatorname{ad} y(z) - \operatorname{ad} y \circ \operatorname{ad} x(z)
\end{aligned}$$

Thus $\operatorname{tr}\operatorname{ad}[x, y] = \operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y - \operatorname{ad} y \circ \operatorname{ad} x) = 0$. Since v is a linear combination of $[x_i, y_i]$ and tr is linear,

$$\operatorname{tr}\operatorname{ad} v = \operatorname{tr}\operatorname{ad}\sum_i a_i[x_i, y_i] = \sum_i a_i \operatorname{tr}\operatorname{ad}[x_i, y_i] = \sum_i a_i 0 = 0$$

□

Proposition 7.10 (Exercise 2.6i)**.** $\mathrm{gl}(2, \mathbb{C}) \cong \mathrm{sl}(2, \mathbb{C}) \oplus \mathbb{C}$

Proof. Let

$$\tilde{\mathbb{C}} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{C} \right\}$$

Then $\mathbb{C} \cong \tilde{\mathbb{C}}$ by the isomorphism

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

We claim that $\mathrm{gl}(2, \mathbb{C}) = \mathrm{sl}(2, \mathbb{C}) \oplus \tilde{\mathbb{C}}$. If we think of $\mathrm{sl}(2, \mathbb{C}) \oplus \tilde{\mathbb{C}}$ not as ordered tuples but as sums of elements from $\mathrm{sl}(2, \mathbb{C})$ and $\tilde{\mathbb{C}}$, then we see that $\mathrm{sl}(2, \mathbb{C}) \oplus \tilde{\mathbb{C}} \subseteq \mathrm{gl}(2, \mathbb{C})$. Then since

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{sl}(2, \mathbb{C})$$

and

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \in \mathrm{sl}(2,\mathbb{C}) \oplus \tilde{\mathbb{C}}$$
$$\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \frac{-1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \in \mathrm{sl}(2,\mathbb{C}) \oplus \tilde{\mathbb{C}}$$

Thus $\mathrm{sl}(2,\mathbb{C})$ contains a basis for $\mathrm{gl}(2,\mathbb{C})$ so it is the entire space. □

Proposition 7.11 (Exercise 2.6ii)**.** *Let L_1, L_2 be Lie algebras. Then*

$$Z(L_1 \oplus Z_2) = Z(L_1) \oplus Z(L_2) \tag{7.1}$$

Proof.

$$\begin{aligned} Z(L_1 \oplus L_2) &= \{(x_1,x_2) : [(x_1,x_2),(y_1,y_2)] = 0 \text{ for all } (y_1,y_2) \in L_1 \oplus L_2\} \\ &= \{(x_1,x_2) : ([x_1,y_1],[x_2,y_2]) = (0,0) \text{ for all } y_1 \in L_1, y_2 \in L_2\} \\ &= \{(x_1,x_2) : x_1 \in Z(L_1), x_2 \in Z(L_2) \\ &= Z(L_1) \oplus Z(L_2) \end{aligned}$$

□

Proposition 7.12 (Exercise 2.6ii)**.** *Let L_1, L_2 be Lie algebras. Then $L_1' \oplus L_2' = (L_1 \oplus L_2)'$.*

Proof. Let $L = L_1 \oplus L_2$.

$$\begin{aligned} L' &= \mathrm{span}\{[x,y] : x,y \in L\} \\ &= \mathrm{span}\{[(x_1,x_2),(y_1,y_2)] : x_1,y_1 \in L_1, x_2,y_2 \in L_2\} \\ &= \mathrm{span}\{([x_1,y_1],[x_2,y_2]) : x_1,y_1 \in L_1, x_2,y_2 \in L_2\} \\ &= \mathrm{span}\{[x_1,y_1] : x_1,y_1 \in L_1\} \oplus \mathrm{span}\{[x_2,y_2] : x_2,y_2 \in L_2\} \\ &= L_1' \oplus L_2' \end{aligned}$$

□

Lemma 7.13 (Lemma for Exercise 2.6ii)**.** *Let L_1, L_2, L_3 be Lie algebras. Then $(L_1 \oplus L_2) \oplus L_3 \cong L_1 \oplus (L_2 \oplus L_3)$*

Proof. The map $((x_1,x_2),x_3) \mapsto (x_1,(x_2,x_3))$ is clearly an isomorphism. □

Proposition 7.14 (Exercise 2.6ii)**.** *If $L = \oplus_{i=1}^k L_i$, then $Z(L) = \oplus_{i=1}^k Z(L_i)$ for $k \in \mathbb{N}$.*

Proof. We have already showed that this is true for $k = 2$ and it is obviously true for $k = 1$. Suppose it is true for $k = n$. Then

$$Z\left(\bigoplus_{i=1}^{n+1} L_i\right) = Z\left(\bigoplus_{i=1}^{n} L_i \oplus L_{n+1}\right) = Z\left(\bigoplus_{i=1}^{n} L_i\right) \oplus Z(L_{n+1}) = \left(\bigoplus_{i=1}^{n} Z(L_i)\right) \oplus Z(L_{n+1}) = \bigoplus_{i=1}^{n+1} Z(L_i)$$

so then it is true for $k = n + 1$. Thus it is true for $k \in \mathbb{N}$. □

Proposition 7.15 (Exercise 2.6ii). *If $L = \oplus_{i=1}^{k} L_i$, then $L' = \oplus_{i=1}^{k} L_i'$ for $k \in \mathbb{N}$.*

Proof. It is obvious for $k = 1$, and we have shown this is true for $k = 2$. Suppose it is true for $k = n$. Then

$$\left(\bigoplus_{i=1}^{n+1} L_i\right)' = \left(\bigoplus_{i=1}^{n} L_i \oplus L_{n+1}\right)' = \left(\bigoplus_{i=1}^{n} L_i\right)' \oplus (L_{n+1})' = \left(\bigoplus_{i=1}^{n} (L_i)'\right) \oplus (L_{n+1})' = \bigoplus_{i=1}^{n+1} (L_i)'$$

so then it is true for $k = n + 1$. Thus it is true for $k \in \mathbb{N}$. □

Proposition 7.16 (Exercise 2.7i). *Let L_1, L_2 be Lie algebras over F. Then*

$$\begin{aligned} p_1 &: L_1 \oplus L_2 \to L_1 \qquad p_1(x_1, x_2) = x_1 \\ p_2 &: L_1 \oplus L_2 \to L_2 \qquad p_2(x_1, x_2) = x_2 \end{aligned}$$

are Lie algebra homomorphisms.

Proof. Let $a, b \in F, x_1, y_1 \in L_1, x_2, y_2 \in L_2$. We show p_1 is linear.

$$p_1\big(a(x_1, x_2) + (y_1, y_2)\big) = p_1(ax_1 + y_1, ax_2 + y_2) = ax_1 + y_1 = ap_1(x_1, x_2) + p_1(y_1, y_2)$$

We show p_1 is bracket-preserving.

$$p_1\big([(x_1, x_2), (y_1, y_2)]\big) = p_1([x_1, y_1], [x_2, y_2]) = [x_1, y_1] = [p_1(x_1, x_2), p_1(y_1, y_2)]$$

We show that p_2 is linear.

$$p_2\big(a(x_1, x_2) + (y_1, y_2)\big) = p_2(ax_1 + y_1, ax_2 + y_2) = ax_2 + y_2 = ap_2(x_1, x_2) + p_2(y_1, y_2)$$

We show that p_2 is bracket-preserving.

$$p_2\big([(x_1, x_2), (y_1, y_2)]\big) = p_2([x_1, y_1], [x_2, y_2]) = [x_2, y_2] = [p_2(x_1, x_2), p_2(y_1, y_2)]$$

Thus p_1, p_2 are Lie algebra homomorphisms. □

Proposition 7.17 (Exercise 2.7i). *Let L_1, L_2 be Lie algebras. Then*

$$\begin{aligned} I_1 &= \{(x_1, 0) : x_1 \in L_1\} \\ I_2 &= \{(0, x_2) : x_2 \in L_2\} \end{aligned}$$

are ideals of $L_1 \oplus L_2$ with $I_1 \cong L_1$ and $I_2 \cong L_2$.

Proof. We showed that p_1, p_2 defined above were Lie algebra homomorphisms. Since $\ker p_1 = I_2$ and $\ker p_2 = I_1$, we know that I_1, I_2 are ideals of $L_1 \oplus L_2$. We define $\phi_1 : I_1 \to L_1$ by $\phi_1(x_1, 0) = x_1$ and $\phi_2 : I_2 \to L_2$ by $\phi_2(0, x_2) = x_2$. ϕ_1, ϕ_2 are easily seen to be isomorphisms. □

Proposition 7.18 (Exercise 2.7ii)**.** *Let L_1, L_2 be Lie algebras with no non-trivial proper ideals. Define I_1, I_2 by*

$$I_1 = \{(x_1, 0) : x_1 \in L_1\}$$
$$I_2 = \{(0, x_2) : x_2 \in L_2\}$$

Let J be a non-trivial proper ideal of $L_1 \oplus L_2$ such that $J \cap I_1 = 0$ and $J \cap I_2 = 0$. Then the projections

$$p_1 : J \to I_1 \qquad p_1(x_1, x_2) = (x_1, 0)$$
$$p_2 : J \to I_2 \qquad p_2(x_1, x_2) = (0, x_2)$$

are isomorphisms.

Proof. We have shown in part (i) that p_1, p_2 are homomorphisms, so we just need to show that they are bijections. Since $J \cap I_1 = 0$ and $J \cap I_2 = 0$, for all $(x_1, x_2) \in J$ either $x_1 = x_2 = 0$ or neither of x_1, x_2 are zero. Thus $\ker p_1 = \ker p_2 = 0$, so p_1, p_2 are one-to-one.

We claim that p_1 is onto. First we show that $\operatorname{im} p_1$ is an ideal of I_1. Let $(x_1, 0) \in \operatorname{im} p_1$ and $(y_1, 0) \in I_1$. Then there exists $(x_1, x_2) \in J$. Then

$$[(x_1, 0), (y_1, 0)] = ([x_1, y_1], [0, 0]) = ([x_1, y_1], 0)$$

Since J is an ideal and $(x_1, x_2) \in J$ and $(y_1, 0) \in L$, $[(x_1, x_2), (y_1, 0)] \in J$. Thus $([x_1, y_1], [x_2, 0]) \in J$, so $p_1([x_1, y_1], [x_2, 0]) = ([x_1, y_1], 0) \in \operatorname{im} p_1$. Thus $\operatorname{im} p_1$ is an ideal of I_1. Since I_1 has no non-trivial proper ideals, $\operatorname{im} p_1 = I_1$, so p_1 is onto.

The same argument works to show that $\operatorname{im} p_2$ is an ideal of I_2 and so p_2 is onto. Thus p_1, p_2 are isomorphisms. □

Lemma 7.19 (Lemma for Exercise 2.7iii)**.** *Let I, J be ideals of L. Then $I \cap J$ is an ideal of L.*

Proof. Let $x \in I \cap J, y \in L$. Then $x \in I$ and $x \in J$ so $[x, y] \in I$ and $[x, y] \in J$ so $[x, y] \in I \cap J$. □

Lemma 7.20. *Let I, J be ideals of L. Then $I \cup J$ is an ideal of L.*

Proof. Let $x \in I \cup J$ and $y \in L$. Then $x \in I$ or $x \in J$, so $[x, y] \in I$ or $[x, y] \in J$. Thus $[x, y] \in I \cup J$. □

Proposition 7.21 (Exercise 2.7iii)**.** *Let L_1, L_2 be non-isomorphic Lie algebras each with no non-trivial proper ideals and let $L = L_1 \oplus L_2$. Then L has exactly two non-trivial proper ideals, which are respectively isomorphic to L_1 and L_2.*

Proof. We have shown in part (ii) that

$$I_1 = \{(x_1, 0) : x_1 \in L_1\}$$
$$I_2 = \{(0, x_2) : x_2 \in L_2\}$$

are ideals of L with $I_1 \cong L_1$ and $I_2 \cong L_2$. Thus we must show that there are no other non-trivial proper ideals of L.

Let J be a non-trivial proper ideal of L with J not equal to I_1 or I_2. If $J \cap I \neq 0$ then $J \cap I_1$ is a non-trivial proper ideal of I_1, but I_1 has no non-trivial proper ideals. Thus $J \cap I_1 = 0$. For analogous reasons, $J \cap I_2 = 0$. Then by part (ii), J is isomorphic to I_1 and I_2, so $J \cong L_1 \cong L_2$. But by hypothesis, $L_1 \ncong L_2$. Thus no such J exists. Thus L has exactly two non-trivial proper ideals. □

Proposition 7.22 (Exercise 2.7iv)**.** *Let L_1 be a one-dimensional Lie algebra over an infinite field F. Let $L_2 \cong L_1$. Then $L = L_1 \oplus L_2$ has infinitely many different ideals.*

Proof. L_1 must be infinite since F is infinite. Let $L_1 = \text{span}\{x\}$. Since L_1 is one-dimensional, all bracket products are zero, so any subalgebra is an ideal. Since F is infinite, it contains a subset isomorphic to $\mathbb{Q}$ which will contain a subset isomorphic to $\mathbb{Z}$. Then the principal ideals

$$< x > = \{nx : n \in \mathbb{Z}\}$$
$$< 2x > = \{n(2x) : n \in \mathbb{Z}\}$$
$$\vdots < kx > = \{n(kx) : n \in \mathbb{Z}\}$$

are infinitey many different ideals of L_1. Thus from any of these we can make an ideal of $L_1 \oplus L_2$ by adding a zero in the L_2 position. □

Proposition 7.23 (Exercise 2.8a)**.** *Let $\phi : L_1 \to L_2$ be an onto homomorphism of Lie algebras. Then $\phi(L_1\prime) = L_2\prime$.*

Proof. By definition, $L_2\prime = \{[x_2, y_2] : x_2, y_2 \in L_2\}$. Since ϕ is onto, for all x,$y_2 \in L_2$ there exist $x_1, y_1 \in L_1$ such that $\phi(x_1) = x_2$ and $\phi(y_1) = y_2$. Thus

$$L_1\prime = \{[\phi(x_1), \phi(y_1)] : x_1, y_1 \in L_1\} = \{[\phi([x_1, y_1]) : x_1, y_1 \in L_1\} = \phi(L_1\prime)$$

□

Proposition 7.24 (Exercise 2.8b)**.** *Let $\phi : L_1 \to L_2$ be an isomorphism. Then $\phi(Z(L_1)) = Z(L_2)$.*

Proof. Let $x \in Z(L_1)$. We need to show that for all $b \in L_2$, $[\phi(x), b] = 0$. Let $b \in L_2$. Then there exists $y \in L_1$ such that $\phi(y) = b$. Since $x \in Z(L_1)$, we know that $[x, y] = 0$, so

$$[\phi(x), b] = [\phi(x), \phi(y)] = \phi([x, y]) = \phi(0) = 0$$

Thus $\phi(Z(L_1) \subseteq Z(L_2)$.

Now suppose $b \in Z(L_2)$. We need to show that there exists $x \in Z(L_1)$ such that $\phi(x) = b$. Since ϕ is onto, there exists $x \in L_1$ such that $\phi(x) = b$. Then for $y \in L_1$, $[\phi(x), \phi(y)] = 0 = \phi([x, y])$. Since ϕ is one-to-one, this implies that $[x, y] = 0$. Thus $x \in Z(L_1)$, so $Z(L_2) \subseteq \phi(Z(L_1))$. □

Proposition 7.25 (Exercise 2.8b)**.** *Let $\phi : L_1 \to L_2$ be an onto Lie algebra homomorphism. Then $Z(L_2)$ is not necesarily contained in $\phi(Z(L_1))$.*

Proof. We provide a specific counterexample. Let L_1 be the Lie algebra

$$\begin{aligned} L_1 &= \operatorname{span}\{z_1, z_2, z_3, x, y\} \\ [z_i, z_j] &= [z_i, x] = [z_j, y] = 0 \\ [x, y] &= x \end{aligned}$$

so $Z(L_1) = \operatorname{span}\{z_1, z_2, z_3\}$ and $L_1' = \operatorname{span}\{x\}$. Let L_2 be the three dimensional abelian Lie algebra spanned by w_1, w_2, w_3. Note that $Z(L_2) = L_2$.

Let $\phi : L_1 \to L_2$ be the linear map defined by

$$\begin{aligned} \phi(x) &= 0 \\ \phi(y) &= w_1 \\ \phi(z_1) &= \phi(z_2) = w_2 \\ \phi(z_3) &= w_3 \end{aligned}$$

Then since $\phi(L_1)$ contains a basis for L_2, ϕ is onto. It is also a homomorphism, because

$$[\phi(a), \phi(b)] = 0\phi([a, b]) \qquad\qquad = \phi(\lambda x) = \lambda\phi(x) = 0$$

for any $a, b \in L_1$, for some $\lambda \in F$. Thus ϕ is a homomorphism. However, $\phi(Z(L_1)) = \operatorname{span}\{w_2, w_2\} \neq Z(L_2) = L_2$. □

Proposition 7.26 (Exercise 2.8c). *Let L_1, L_2 be Lie algebras over F. If $\phi : L_1 \to L_2$ is an isomorphism and $x \in L_1$ such that $\operatorname{ad} x$ diagonlisable, then $\operatorname{ad} \phi(x)$ is diagonlisable.*

Proof. Let $\beta = \{v_i\}_{i=1}^n$ be a basis for L_1 so that $\operatorname{ad} x$ is diagonlisable with respect to β. Then $\operatorname{ad} x(v_i) = [x, v_i] = \lambda_i v_i$ for $\lambda_i \in F$. Then $\phi(\beta)$ is a basis for L_2, and

$$\operatorname{ad} \phi(x)(\phi(v_i)) = [\phi(x), \phi(v_i)] = \phi([x, v_i]) = \phi(\lambda_i v_i) = \lambda_i \phi(v_i)$$

so $\operatorname{ad} \phi(x)$ is diagonlisable with respect to $\phi(\beta)$. □

Proposition 7.27 (Exercise 2.8c). *Let L_1, L_2 be Lie algebras over F, and let $\phi : L_1 \to L_2$ be an onto homomorphism. Let $x \in L_1$ such that $\operatorname{ad} x$ is diagonlisable. Then $\operatorname{ad} \phi(x)$ is diagonalisable.*

Proof. Let $\beta = \{v_i\}_{i=1}^n$ be a basis for L_1 so that $\operatorname{ad} x$ is diagonlisable with respect to β. Then $\operatorname{ad} x(v_i) = [x, v_i] = \lambda_i v_i$ for $\lambda_i \in F$. Since ϕ is an onto homomorphism, $\phi(\beta)$ is a spanning set for L_2. Then $\phi(\beta)$ contains a basis for L_2, denote this basis by γ, where $\gamma \subseteq \phi(\beta)$. Define $\beta' = \phi^{-1}(\gamma)$. Then for all $v_i \in \beta'$ (equivalently for each $\phi(v_i) \in \gamma$),

$$\operatorname{ad} \phi(x)(\phi(v_i)) = [\phi(x), \phi(v_i)] = \phi([x, v_i]) = \phi(\lambda_i v_i) = \lambda_i \phi(v_i)$$

Thus $\operatorname{ad} \phi(x)$ is diagonlisable with respect to γ. □

Proposition 7.28 (Exercise 2.9). $\mathbb{R}^3_\wedge \cong L = \{x \in \operatorname{gl}(3, \mathbb{R}) : x^t = -x\}$.

Proof. We know that $i = (1,0,0), j = (0,1,0), k = (0,0,1)$ forms a basis for $\mathbb{R}^3_\wedge$ with

$$[i,j] = k \qquad [j,k] = i \qquad [k,i] = j$$

We also know that

$$e = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

forms a basis for L with

$$[e,f] = g \qquad [f,g] = e \qquad [g,e] = f$$

Thus $\phi : \mathbb{R}^3_\wedge \to L$ defined by $\phi(i) = e, \phi(j) = f, \phi(k) = h$ is an isomorphism. □

Proposition 7.29 (Exercise 2.9)**.** *Let*

$$L_1 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a,b,c \in \mathbb{R} \right\}$$
$$L_2 = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a,b,c \in \mathbb{R} \right\}$$

then $L_1 \not\cong L_2$ *as Lie algebras.*

Proof. First we claim that $L_2' \subseteq Z(L_2)$. Let $A, B \in L_2$ be

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & e & f \\ 0 & 0 & g \\ 0 & 0 & 0 \end{pmatrix}$$

We compute the bracket product of A and B:

$$[A,B] = AB - BA = \begin{pmatrix} 0 & 0 & ag - ce \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus

$$L_2' = \operatorname{span} \left\{ C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

Then for $A \in L_2$, $C \in L_2'$, we compute $[A,C] = 0$. Thus $C \in Z(L)$, so $L_2' \subseteq Z(L_2)$.

Now we claim that $L_1' \not\subseteq Z(L_1)$. We know that

$$L_1' = \operatorname{span} \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

but we compute

$$\left[\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right] = \begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix} \neq \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

so

$$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \notin Z(L_1)$$

Thus $L_1' \not\subseteq Z(L_1)$.

Suppose $\phi : L_1 \to L_2$ were an isomorphism. Then ϕ would preserve the property of the derived algebra being contained in the center, but since these algebras do not share this property, no such isomorphism exists. Thus $L_1 \not\cong L_2$. $\square$

To summarize our results for Exercise 2.9, we showed that (i) and (iv) are isomorphic and that (ii) and (iii) are not isomorphic. It is easy to see that (i) is not isomorphic to (ii) or (iii), since the derived algebra for (i) has dimension 3 and the derived algebras for (ii) and (iii) have dimension 1.

Proposition 7.30 (Exercise 2.10)**.** *Let F be a field. Then* $\mathrm{gl}(n, F)' = \mathrm{sl}(n, F)$*.*

Proof. Since $[x, y] = xy - yx$ has trace zero for any $x, y \in \mathrm{gl}(n, F)$, clearly $\mathrm{gl}(n, F)' \subseteq \mathrm{sl}(n, F)$. To show equality, we will show that $\mathrm{gl}(n, F)'$ contains a basis for $\mathrm{sl}(n, F)$. This makes it a subspace of equal dimension.

A basis for $\mathrm{sl}(n, F)$ is given by e_{ij} for $i \neq j$ and $e_{ii} - e_{i+1,i+1}$ for $1 \leq i < n$, as stated on page 3 of Erdmann and Wildon. Using the formula for bracket products of e_{ij} also on page 3, we compute

$$\begin{aligned} [e_{i1}, e_{1j}] &= \delta_{11} e_{ij} = e_{ij} \text{ for } i \neq j \\ [e_{i,i+1}, e_{i+1,i}] &= \delta_{i+1,i+1} e_{ii} - \delta_{ii} e_{i+1,i+1} = e_{ii} - e_{i+1,i+1} \text{ for } 1 \leq i < n \end{aligned}$$

Thus $\mathrm{gl}(n, F)'$ contains this basis for $\mathrm{sl}(n, F)$, so it is a subspace of equal dimension, so $\mathrm{gl}(n, F)' = \mathrm{sl}(n, F)$. $\square$

Proposition 7.31 (Exercise 2.11)**.** *Let $S \in \mathrm{gl}(n, F)$ and let P be an invertible matrix in $\mathrm{gl}(n, F)$. Let $A = P^T S P$. Then $\mathrm{gl}_A(n, F) \cong \mathrm{gl}_S(n, F)$ (Lie algebra isomorphism).*

Proof. Define $\phi : \mathrm{gl}_S(n, F) \to \mathrm{gl}_A(n, F)$ by $\phi(x) = P^{-1} x P$. First we show that ϕ actually maps into $\mathrm{gl}_A(n, F)$, so we need to show that for $x \in \mathrm{gl}_S(n, F)$, $\phi(x) \in \mathrm{gl}_A(n, F)$. Let $x \in \mathrm{gl}_S(n, F)$. Then

$$\begin{aligned} x^T S &= -Sx \\ P^T x^T S &= -P^T S x \\ P^T x^T S P &= -P^T S x P \\ P^T x^T (P^T)^{-1} P^T S P &= -P^T S P P^{-1} x P \\ \phi(x)^T P^T S P &= -P^T S P \phi(x) \\ \phi(x)^T A &= -A \phi(x) \end{aligned}$$

thus $\phi(x) \in \mathrm{gl}_A(n, F)$.

Now we show that ϕ is one-to-one. Let $x, y \in \mathrm{gl}_S(n, F)$ with $x = y$. Then

$$\begin{aligned} P^{-1}x &=^{-1} y \\ P^{-1}xP &= P^{-1}yP \\ \phi(x) &= \phi(y) \end{aligned}$$

thus ϕ is one-to-one.

Now we show that ϕ is onto. Let $z \in \mathrm{gl}_A(n, F)$. We claim that $PzP^{-1} \in \mathrm{gl}_S(n, F)$ and that $\phi(PzP^{-1}) = z$. Since $z \in \mathrm{gl}_A(n, F)$,

$$\begin{aligned} z^T P^T SP &= -P^T SPz \\ (P^{-1})^T z^T p^T SP &= -SPz \\ (PzP^{-1})^T SP &= -SPz \\ (PzP^{-1})^T S &= -S(PzP^{-1}) \end{aligned}$$

thus $PzP^{-1} \in \mathrm{gl}_S(n, F)$. Then $\phi(PzP^{-1}) = P^{-1}PzP^{-1}P = z$. Thus ϕ is onto.

Finally, we show that ϕ is a homomorphism. Let $x, y \in \mathrm{gl}_S(n, F)$. Then

$$\begin{aligned} \phi([x, y]) &= P^{-1}(xy - yx)P \\ &= P^{-1}xyP - P^{-1}yxP \\ &= P^{-1}xyPP^{-1}yP - P^{-1}yPP^{-1}xP \\ &= \phi(x)\phi(y) - \phi(y)\phi(x) \\ &= [\phi(x), \phi(y)] \end{aligned}$$

thus ϕ preserves the bracket. Thus we have shown that ϕ is a bijection and a homomorphism, so ϕ is an isomorphism. □

Proposition 7.32 (Exercise 2.12)**.** *Let S be an $n \times n$ intvertible matrix with entries in $\mathbb{C}$. Then for $x \in \mathrm{gl}_S(n, \mathbb{C}), \operatorname{tr} x = 0$.*

Proof. Let $x \in \mathrm{gl}_S(n, \mathbb{C}) = \{y \in \mathrm{gl}(n, \mathbb{C}) : y^T S = -Sy\}$. Then

$$\begin{aligned} x^T S &= -Sx \\ x &= -S^{-1}x^T S \end{aligned}$$

So then the traces are equal,

$$\operatorname{tr} x = \operatorname{tr}(-S^{-1}x^T S) = -\operatorname{tr}(S^{-1}x^T S) = -\operatorname{tr}(x^T S^{-1} S) = -\operatorname{tr}(x^T) = -\operatorname{tr} x$$

Thus $\operatorname{tr} x = -\operatorname{tr} x$, and since $\operatorname{tr} x \in \mathbb{C}$, we must have $\operatorname{tr} x = 0$. □

Proposition 7.33 (Exercise 2.13)**.** *Let I be an ideal of the Lie algebra L over field F. Then*

$$B = C_L(I) = \{x \in L : [x, a] = 0 \text{ for } a \in I\}$$

is an ideal of L.

Proof. First we show that B is closed under addition in L. Let $b_1, b_2 \in B$ and let $a \in I$. Then

$$[b_1 + b_2, a] = [b_1, a] + [b_2, a] = 0 + 0 = 0$$

thus $b_1 + b_2 \in B$.

Now we show that B is closed under scalar multiplication from F. Let $\lambda \in F$ and $b \in B$ and $a \in I$. Then

$$[\lambda b, a] = \lambda[b, a] = 0$$

thus $\lambda b \in B$.

Now we show that for $b \in B$, $y \in L$, $[y, b] \in B$. Let $a \in I$. Since I is an ideal of L, $[a, y] \in I$, so by definition of B, $[b, a] = 0$ and $[b, [a, y]] = 0$. Then by the Jacobi identity,

$$[a, [y, b]] + [y, [b, a]] + [b, [a, y]] = 0$$

Since $[b, a] = 0$ and $[b, [a, y]] = 0$, the second and third terms are zero. Thus $[a, [y, b]] = 0$, so $[[y, b], a] = 0$ so by definition of B, $[y, b] \in B$. Thus B is an ideal of L. □

Proposition 7.34 (Exercise 2.14i)**.** *Let L be all matrices*

$$L = \left\{ \begin{pmatrix} 0 & f(x) & h(x,y) \\ 0 & 0 & g(y) \\ 0 & 0 & 0 \end{pmatrix} : f(x) \in R[x], g(y) \in R[y], h(x,y) \in R[x,y] \right\}$$

L is a Lie algebra (over $\mathbb{R}$) with the bracket $[a, b] = ab - ba$.

Proof. Clearly L is closed under matrix addition and multiplication. We know that matrix multiplication and addition are bilinear. Furthermore, it is obvious that $[a, a] = a^2 - a^2 = 0$. Now we need to show the Jacobi identity holds. Let $a, b, c \in L$.

$$\begin{aligned} [a, [b, c]] + [b, [c, a]] + [c, [a, b]] &= a(bc - bc) - (bc - cb)a + b(ca - ac) \\ &\quad - (ca - ac)b + c(ab - ba) - (ab - ba)c \\ &= abc - acb - bca + cba + bca - bac \\ &\quad - cab + acb + cab - cba - abc + bac \\ &= 0 \end{aligned}$$

□

Proposition 7.35 (Exercise 2.14ii)**.** *Let A, B be matrices in L where*

$$L = \left\{ \begin{pmatrix} 0 & f(x) & h(x,y) \\ 0 & 0 & g(y) \\ 0 & 0 & 0 \end{pmatrix} : f(x) \in R[x], g(y) \in R[y], h(x,y) \in R[x,y] \right\}$$

Then

$$[A, B] = \begin{pmatrix} 0 & 0 & f_A g_B - f_B g_A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

thus

$$L' = \text{span}\{\begin{pmatrix} 0 & 0 & f_A g_B - f_B g_A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : f_A, f_B \in R[x], g_A, g_B \in R[y]\}$$

Proof. Let

$$A = \begin{pmatrix} 0 & f_A & h_A \\ 0 & 0 & g_A \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & f_B & h_B \\ 0 & 0 & g_B \\ 0 & 0 & 0 \end{pmatrix}$$

Then we compute $[A, B]$ as

$$[A, B] = AB - BA = \begin{pmatrix} 0 & 0 & f_A g_B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & f_B g_A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & f_A g_B - f_B g_A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

□

8 Chapter 3 Exercises

Proposition 8.1 (Page 21, from 3.2.1)**.** *Let L be a 3-dimensional Lie algebra over F with $L' \subseteq Z(L)$ and L' of dimension one. Let $f, g \in L$ such that $[f, g] \neq 0$. Then let $z = [f, g]$. Then f, g, z are linearly independent, and thus form a basis for L.*

Proof. Supose $af + bg + cz = 0$ for some $a, b, c \in F$. Then for all $x \in L$, $[af + bg + cz, x] = 0$, so $a[f, x] + b[g, x] + c[z, x] = 0$. Since $z \in L' \subseteq Z(L)$, we know that $[z, x] = 0$. Thus $a[f, x] + b[g, x] = 0$. In particular, this is true for $x = f$, so $a[f, f] + b[g, f] = 0$ so $b[g, f] = 0$, Since $[f, g] \neq 0$ by hypothesis, we conclude that $b = 0$. Likewise, when we set $x = g$, we see that $a[f, g] = 0$, so $a = 0$. Then returning to the original equation, we see that $cz = 0$, which implies that $c = 0$ since $z \neq 0$ by hypothesis. Thus $a = b = c = 0$, so f, g, z are linearly independent and form a basis for L. □

Proposition 8.2 (Exercise 3.1)**.** *Let $V = \operatorname{span}\{v_1, \ldots v_2\}$ be a vector space and let $\phi : V \to V$ be a linear map. let $L = V \oplus \operatorname{span}\{x\} = \operatorname{span}\{v_1, \ldots v_n, x\}$ and define a bracket*

$$[,] : L \times L \to L$$

as the bilinear map defined by

$$[v_i, v_j] = 0 \qquad [x, x] = 0 \qquad [x, v_i] = \phi(v_i)$$

Then L is a Lie algebra under this bracket with $\dim L' = \operatorname{rank} \phi$.

Proof. We know that since $[y, y] = 0$ for all $y \in L$ that the bracket is antisymmetric, as shown in pages 1-2 of Erdmann and Wildon. We just need to show that the Jacobi identity holds for any three basis elements. If any two are in the span of x, then all brackets will be zero, and if all are in V, then the brackets will be zero. So we just need to show the identity holds in the case of x and two basis vectors v_i, v_j of V.

$$[x, [v_i, v_j]] + [v_i, [v_j, x]] + [v_j, [x, [v_i]] = [x, 0] - [v_i, \phi(v_j)] + [v_j, \phi(v_i)] = 0 - 0 + 0 = 0$$

Thus L is a Lie algebra under this bracket. Furthermore,

$$L' = \operatorname{span}\{[y, z] : y, z \in L\} = \operatorname{span}\{\phi(v_i) : 1 \leq i \leq n\} = \phi(V)$$

Thus $\dim L' = \dim \phi(V)$, and $\operatorname{rank} \phi = \dim \phi(V)$ by definition. □

Proposition 8.3 (Exercise 3.2)**.** *Let L_u, L_v be 3-dimensional Lie algebras over $\mathbb{C}$ with respective bases $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$ where $L'_u = \operatorname{span}\{y_1, z_1\}$ and $L'_v = \operatorname{span}\{y_2, z_2\}$ and $\operatorname{ad} x_1 : L'_u \to L'_u$ and $\operatorname{ad} x_2 : L'_v \to L'_v$ are diagonalisable with matrices*

$$[\operatorname{ad} x_1] = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \qquad [\operatorname{ad} x_2] = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$$

with respect to the aforementioned bases, where $u, v \in \mathbb{C}$ with $u, v \neq 0$. If $v = u$ or $v = u^{-1}$ then $L_u \cong L_v$.

Proof. Note that Erdmann and Wildon prove in 3.2.3 that L'_u, L'_v are abelian. From the matrices $[\operatorname{ad} x_1]$ and $[\operatorname{ad} x_2]$ we know that

$$\begin{aligned} [x_1, y_1] &= y_1 & \qquad [x_2, y_2] &= y_2 \\ [x_1, z_1] &= uz_1 & \qquad [x_2, z_2] &= vz_2 \\ [y_1, z_1] &= 0 & \qquad [y_2, z_2] &= 0 \end{aligned}$$

Suppose that $u = v$. We define the linear map $\phi : L_u \to L_v$ by

$$\begin{aligned} \phi(x_1) &= x_2 \\ \phi(y_1) &= y_2 \\ \phi(z_1) &= z_2 \end{aligned}$$

Since ϕ maps a basis to a basis, it is bijective. We need to show that ϕ also preserves brackets. Thus we compute how the brackets interact with ϕ as follows:

$$\begin{aligned} \phi([x_1, y_1]) &= \phi(y_1) = y_2 = [x_2, y_2] = [\phi(x_1), \phi(y_1)] \\ \phi([x_1, z_1]) &= \phi(uz_1) = uz_2 = vz_2 = [x_2, z_2] = [\phi(x_1), \phi(z_2)] \\ \phi([y_1, z_1]) &= \phi(0) = 0 = [y_2, z_2] = [\phi(y_1), \phi(z_2)] \end{aligned}$$

Thus ϕ is an isomorphism of Lie algebras. Now suppose that $v = u^{-1}$. We define a linear map $\psi : L_u \to L_v$ by

$$\begin{aligned} \psi(x_1) &= ux_2 \\ \psi(y_1) &= z_2 \\ \psi(z_1) &= y_2 \end{aligned}$$

Since ux_2 is just a scalar multiple of x_2, $\{ux_2, z_2, y_2\}$ is a basis for L_v, so ψ maps a basis to a basis, so it is a bijection. We need to show that it also preserves brackets. Note that $x_2 = u^{-1}\psi(x_1)$.

$$\begin{aligned} \psi([x_1, y_1]) &= \psi(y_1) = z_2 = uu^{-1}z_2 = u[x_2, z_2] = u[u^{-1}\psi(x_1), \psi(y_1)] = [\psi(x_1), \psi(y_1)] \\ \psi([x_1, z_1]) &= \psi(uz_1) = u\psi(z_1) = uy_2 = u[x_2, y_2] = u[u^{-1}\psi(x_1), \psi(z_1)] = [\psi(x_1), \psi(z_1)] \\ \psi([y_1, z_1]) &= \psi(0) = 0 = [z_2, y_2] = [\psi(y_1), \psi(z_1)] \end{aligned}$$

Thus ψ is an isomorphism of Lie algebras. □

Proposition 8.4 (Exercise 3.2). *Let L_u, L_v be 3-dimensional Lie algebras over $\mathbb{C}$ with respective bases $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2$ where $L'_u = \operatorname{span}\{y_1, z_1\}$ and $L'_v = \operatorname{span}\{y_2, z_2\}$ and $\operatorname{ad} x_1 : L'_u \to L'_u$ and $\operatorname{ad} x_2 : L'_v \to L'_v$ are diagonalisable with matrices*

$$[\operatorname{ad} x_1] = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \qquad [\operatorname{ad} x_2] = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$$

with respect to the aforementioned bases, where $u, v \in \mathbb{C}$ with $u, v \neq 0$. If $L_u \cong L_v$, then $v = 0$ or $v = u^{-1}$.

Proof. Suppose $\phi : L_u \to L_v$ is an isomorphism. By Exercise 2.8a, when restricted to L'_u, $\phi : L'_u \to L'_v$ is still an isomorphism. Since ϕ is onto, one of the basis elements of L_u must map to a linear combination that has a non-zero scalar for x_2, and since $\phi(L'_u) = L'_v$, that basis element must be x_1. In particular, it must be the case that $\phi(x_1) = ax_2 + w$ for some non-zer $a \in \mathbb{C}$ and some $w \in L'_v$.

Now let $t \in L'_u$. We can compute

$$[\phi(x_1), \phi(t)] = \phi([x_1, t]) = \phi \circ \operatorname{ad} x_1(t)$$
$$[\phi(x_1), \phi(t)] = [ax_2 + w, \phi(t)] = a[x_2, \phi(t)] + [w, \phi(t)] = a[x_2, \phi(t)] + 0 = a \operatorname{ad} x_2 \circ \phi(t)$$

Note that the $[w, \phi(t)]$ term is zero because $w, \phi(t) \in L'_v$ and L'_v is abelian as shown in Lemma 3.3a of Erdmann and Wildon. From this we see that $\phi \circ \operatorname{ad} x_1 = a \operatorname{ad} x_2 \circ \phi = \operatorname{ad}(ax_2) \circ \phi$. Let $[\phi]$ denote the matrix of ϕ. Since ϕ a bijection, it is invertible, so the matrix $[\phi]$ is invertible. As shown in 16.1i, the matrix of a composition is the product of the matrices, so

$$[\phi \circ \operatorname{ad} x_1] = [\phi][\operatorname{ad} x_1]$$
$$[\operatorname{ad}(ax_2) \circ \phi] = [\operatorname{ad}(ax_2)][\phi]$$

Since the maps $\phi \circ \operatorname{ad} x_1$ and $\operatorname{ad}(ax_2) \circ \phi$ are equal, their matrices are equal, so

$$\begin{aligned} &[\phi \circ \operatorname{ad} x_1] = [\operatorname{ad}(ax_2) \circ \phi] \\ \Longrightarrow\ &[\phi][\operatorname{ad} x_1] = [\operatorname{ad}(ax_2)][\phi] \\ \Longrightarrow\ &[\phi][\operatorname{ad} x_1][\phi]^{-1} = [\operatorname{ad}(ax_2)] \end{aligned}$$

Thus the matrices for $\operatorname{ad} x_1 : L_u \to L_u$ and $\operatorname{ad}(ax_2) : L_v \to L_v$ are similar, so they are similar as linear maps. In particular, this means that they have the same eigenvalues. The eigenvalues for $\operatorname{ad} x_1$ are $\{1, u\}$ and the eigenvalues for $\operatorname{ad}(ax_2)$ are $\{a, av\}$, so we have $\{1, u\} = \{a, av\}$. Thus either $a = 1$ and $u = v$, or $a = u$ and $v = u^{-1}$. This completes the proof, since we have shown that $u = v$ or $u = v^{-1}$. □

Proposition 8.5 (Exercise 3.3i)**.** *Let*

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Then $\mathrm{gl}_S(3, \mathbb{C}) \cong \mathrm{sl}(2, \mathbb{C})$.

Proof. Let $x \in \mathrm{gl}_S(3, \mathbb{C})$. Then $x^t S = -Sx$, so

$$x = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ b & c & 0 \end{pmatrix}$$

for some $a, b, c \in \mathbb{C}$. Thus a basis for $\mathrm{gl}_S(3, \mathbb{C})$ is

$$e = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

where the bracket products compute to

$$[e, f] = g \qquad [f, g] = e \qquad [e, g] = f$$

which is clearly isomorphic to the $\mathrm{sl}(2, \mathbb{C})$ by mapping this basis $\{e, f, g\}$ to the basis $\{e, f, g\}$ described in Exercise 1.12.

Alternately, even without this explicit isomorphism, we can deduce this isomorphism from the theorem that there is only one Lie algebra over $\mathbb{C}$ with $\dim L' = 3$. □

Proposition 8.6 (Exercise 3.3ii). *Let L b e the complex matrix algebra spanned by*

$$U = \begin{pmatrix} t & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & v \end{pmatrix} \qquad V = e_{13} \qquad W = e_{23}$$

for some fixed $t, u, v \in \mathbb{C}$. Then in the notation of 3.2.3, L is isomorphic to L_x where $x = \frac{u-v}{1-v}$.

Proof. We can compute the brackets

$$\begin{aligned} [U, V] &= (1 - v)V \\ [U, W] &= (u - v)W[V, W] & = 0 \end{aligned}$$

Thus L' has basis V, W so $\dim L' = 2$. Furthermore, the map $\operatorname{ad} U : L' \to L'$ is diagonalisable, with matrix

$$M_{\operatorname{ad} U} = \begin{pmatrix} 1 - v & 0 \\ 0 & u - v \end{pmatrix}$$

Let $U_2 = (1 - v)^{-1}U$. Then U_2, V, W is still a basis for L, and now we get the bracket products

$$\begin{aligned} [U_2, V] &= [(1 - v)^{-1}U, V] = (1 - v)^{-1}[U, V] = V \\ [U_2, W] &= (1 - v)^{-1}[U, W] = \left(\frac{u - v}{1 - v}\right) W \end{aligned}$$

so the matrix of U_2 is

$$M_{\operatorname{ad} U_2} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{u-v}{1-v} \end{pmatrix}$$

Thus in the notation of 3.2.3, we have that L is isomorphic to L_x with $x = \frac{u-v}{1-v}$. □

Proposition 8.7 (Exercise 3.3iii). *Let L be the complex matrix Lie algebra*

$$L = \left\{ \begin{pmatrix} 0 & a & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{where } a, b, c \in \mathbb{C} \right\}$$

Then L is isomorphic to the Heisenberg algebra.

Proof. First we show that L' is one-dimensional. Let

$$A = \begin{pmatrix} 0 & a & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & d & e & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then we compute $[A, B]$.

$$[A, B] = AB - BA = \begin{pmatrix} 0 & 0 & af - cd & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so L' is one-dimensional, since

$$C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is a basis for L'. Then we can see that C is in $Z(L)$ and thus $L' \subset Z(L)$ becasuse $[A, C] = 0$ and A is a general matrix from L. Thus L is a three dimensional Lie algebra with $\dim L' = 1$ and $L' \subset Z(L)$, so it is the Heisenberg Algebra. □

Proposition 8.8 (Exercise 3.3iv)**.** *Let L be the complex matrix Lie algebra*

$$L = \left\{ \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ where } a, b, c \in \mathbb{C} \right\}$$

Then L is abelian.

Proof. Let

$$A = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then we compute $[A, B] = 0$. Thus L is abelian. □

Proposition 8.9 (Exercise 3.4)**.** *Let L be a vector space over F with basis v_1, v_2 and bilinear operator $[,] : L \times L \to L$ with $[u, u] = 0$ for $u \in L$. Then the Jacobi identity holds for this bilinear operator, and so L is a Lie algebra with this bracket.*

Proof. First note that since $[u, u] = 0$ for all $u \in L$, it follows that the bracket is anti-commutative (see page 1 of Erdmann and Wildon for proof). Let $x, y, z \in L$. There are two possibilities: x, y are linearly independent or they are not. If they are not linearly independent, then $x = \lambda y$ for some $\lambda \in F$. Then

$$\begin{aligned}
[x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= [\lambda y,[y,z]] + [y,[z,\lambda y]] + [z,[\lambda y, y]] \\
&= \lambda[y,[y,z]] - \lambda[y,[y,z]] + 0 \\
&= 0
\end{aligned}$$

where the third term goes to zero because $[y, y] = 0$. If x, y are linearly independent, then they form a basis for L. Then we can write $z = ax + by$ for some $a, b \in F$. Then

$$\begin{aligned}
[x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= [x, a[y,x]] + [y, b[y,x]] + [ax,[x,y]] + [by,[x,y]] \\
&= a[x,[y,x]] + b[y,[y,x]] + a[x,[x,y]] + b[y,[x,y]] \\
&= a[x,[y,x]] - a[x,[y,x]] + b[y,[y,x]] - b[y,[y,x]] \\
&= 0
\end{aligned}$$

□

Proposition 8.10 (Exercise 3.5)**.** *There exists* $h \in \mathrm{sl}(2,\mathbb{R})$ *such that* $\mathrm{ad}\, h$ *is diagonalisable.*

Proof. Let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This is a basis for $\mathrm{sl}(2,\mathbb{R})$. The bracket products are

$$[h, e] = 2e \qquad [h, f] == 2f \qquad [h, h] = 0$$

so the matrix of $\mathrm{ad}\, h$ with respect to the basis e, f, h is

$$M_{\mathrm{ad}\, h} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is clearly a diagonal matrix. Thus $\mathrm{ad}\, h$ is diagonalisable. □

Proposition 8.11 (Exercise 3.5)**.** *There is no* $x \in \mathbb{R}^3_\wedge$ *with* $x \neq 0$ *such that* $\mathrm{ad}\, x$ *is diagonalisable.*

Proof. Let u, v, w be a basis for $\mathbb{R}^3_\wedge$ and suppose that there is an $x \neq 0$ such that $\mathrm{ad}\, x$ is diagonalisable. Then for some $a, b, c \in \mathbb{R}$,

$$\begin{aligned}
x \times u &= au \\
x \times v &= bv \\
x \times w &= cw
\end{aligned}$$

But if $x \neq 0$, then $x \times u$ is orthogonal to u, so $x \times u$ cannot be collinear with u unless $x = 0$ or $u = 0$. But $x \neq 0$ by hypothesis and $u \neq 0$ since u is part of a basis. Thus no such x exists. □

Proposition 8.12 (Exercise 3.5). *Let L_1, L_2 be Lie algebras over F. If $\phi : L_1 \to L_2$ is an isomorphism and $x \in L_1$ such that* $\operatorname{ad} x$ *diagonlisable, then* $\operatorname{ad} \phi(x)$ *is diagonlisable.*

Proof. Let $\beta = \{v_i\}_{i=1}^n$ be a basis for L_1 so that $\operatorname{ad} x$ is diagonlisable with respect to β. Then $\operatorname{ad} x(v_i) = [x_i, v_i] = \lambda_i v_i$ for $\lambda_i \in F$. Then $\phi(\beta)$ is a basis for L_2, and

$$\operatorname{ad} \phi(x)(\phi(v_i)) = [\phi(x), \phi(v_i)] = \phi([x, v_i]) = \phi(\lambda_i v_i) = \lambda_i \phi(v_i)$$

so $\operatorname{ad} \phi(x)$ is diagonlisable with respect to $\phi(\beta)$. □

Proposition 8.13 (Exercise 3.5). $\mathrm{sl}(2, \mathbb{R}) \ncong \mathbb{R}^3_\wedge$

If $\phi : \mathrm{sl}(2, \mathbb{R}) \to \mathbb{R}^3_\wedge$ were an isomorphism, then $\operatorname{ad} \phi(h)$ would be diagonlisable. However, there is no $x \in \mathbb{R}^3_\wedge$ with $\operatorname{ad} x$ diagonlisable. Thus there is no such isomorphism ϕ.

Proposition 8.14 (Exercise 3.7). *Let L be a non-abelian Lie algebra. Then* $\dim Z(L) \leq \dim L - 2$.

Proof. Clearly it is impossible for $\dim Z(L)$ to be greater than or equal to $\dim L$, since $Z(L)$ is a proper ideal of L. So, to prove our claim, all we need to do is rule out the possibility that $\dim Z(L) = \dim L - 1$.

Let $n = \dim L$. Suppose that $\dim Z(L) = n - 1$. Then we have a basis $\{v_1, v_2, \ldots v_{n-1}\}$ for $Z(L)$. We can extend this to a basis of L by appending the vector $u \in L$, so we have a basis $\{v_1, v_2, \ldots v_{n-1}, u\}$ for L. Then for all $1 \leq i \leq n$, $[v_i, u] = 0$ since $v_i \in Z(L)$, and for all $1 \leq i, j \leq n$, $[v_i, v_j] = 0$. Thus all bracket products of basis elements of L are zero, so L is abelian. This contradicts our hypothesis that L is non-abelian, so we conclude that it is impossible for $\dim Z(L) = \dim L - 1$. □

Proposition 8.15 (Exercise 3.9i). *Let L be a Lie algebra with an ideal I and subalgebra S such that $L = S \oplus I$. Let $\theta : S \to \mathrm{gl}(I)$ be defined by $\theta(s)(x) = [s, x]$. Then θ is a Lie algebra homomorphism from S into* $\operatorname{Der} I$.

Proof. Bilinearity of θ follows from bilinearity of the bracket on L:

$$\theta(s)(ax + b) = [s, ax + b] = a[s, x] + [s, y] = a\theta(s)(x) + \theta(s)(y)$$

We claim that θ preserves the bracktes on S and $\mathrm{gl}(I)$, that is, for $s, t \in S$, $\theta([s, t]) = [\theta(s), \theta(t)]$. Let $x \in I$. Then

$$\begin{aligned}
\theta([s, t])(x) &= [[s, t], x] \\
&= -[x, [s, t]] \\
&= [s, [t, x]] + [t, [x, s]] && \text{by Jacobi} \\
&= [s, [t, x]] - [t, [s, x]] \\
&= (\theta(s) \circ \theta(t))(x) - (\theta(t) \circ \theta(s))(x) \\
&= [\theta(s), \theta(t)](x)
\end{aligned}$$

Thus θ is a Lie algebra homomorphism. We also claim that $\operatorname{im} \theta \subseteq \operatorname{Der} I$. Let $s \in S$. Then for $x, y \in I$,

$$\theta(s)[x, y] = [s, [x, y]] = -[x, [y, s]] - [y, [s, x]] = [x, [s, y]] + [[s, x], y] = [x, \theta(s)(y)] + [\theta(s)(x), y]$$

Thus $\theta(s)$ is a derivation of I, so $\operatorname{im} \theta \subseteq \operatorname{Der} I$. □

Proposition 8.16 (Exercise 3.9ii)**.** *Let S, I be Lie algebras over F and let $\theta : S \to \operatorname{Der} I$ be a Lie algebra homomorphism. We equip the vector space $S \oplus I$ with the bracket*

$$[(s_1, x_1), (s_2, x_2)] = ([s_1, s_2], [x_1, x_2] + \theta(s_1)x_2 - \theta(s_2)x_1)$$

We claim that $S \oplus I$ is a Lie algebra under this bracket.

Proof. First we show that this bracket is bilinear. Let $a, b \in F$ and let $(s_1, x_1), (s_2, x_2), (s_3, x_3) \in S \oplus I$.

$$\begin{aligned}
[a(s_1, x_1) + b(s_2, x_2), (s_3, x_3)] &= [(as_1 + bs_2, ax_1 + bx_2), (s_3, x_3)] \\
&= \big([as_1 + bs_2, s_3], \\
&\qquad [ax_1 + bx_2, x_3] + \theta(as_1 + bs_2)(x_3) - \theta(s_3)(ax_1 + bx_2)\big) \\
&= \big(a[s_1, s_3] + b[s_2, s_3], \\
&\qquad a[x_1, x_3] + b[x_1, x_3] + a\theta(s_1)x_3 + b\theta(s_2)x_3 - a\theta(s_3)x_1 - b\theta(s_3)x_3\big) \\
&= a([s_1, s_3], [x_1, x_3] - \theta(s_1)x_3 - \theta(s_3)x_1) \\
&\qquad b([s_2, s_3], [x_1, x_3] + \theta(s_2)x_3 - \theta(s_3)x_2 \\
&= a[(s_1, x_1), (s_3, x_3)] + b[(s_2, x_2), (s_3, x_3)]
\end{aligned}$$

Thus the bracket is linear in the first entry. Now we show linearity of the bracket in the second entry.

$$\begin{aligned}
[(s_1, x_1), a(s_2, x_2) + b(s_3, x_3)] &= [(s_1, x_1), (as_2 + bs_3, ax_2 + bx_3)] \\
&= \big([s_1, as_2 + bs_3], [x_1, ax_2 + bx_3] + \theta(s_1)(ax_2 + bx_3) - \theta(as_2 + bs_3)x_1\big) \\
&= \big(a[s_1, s_2] + b[s_1, s_3], \\
&\qquad a[x_1, x_2] + b[x_1, x_3] + a\theta(s_1)x_2 + b\theta(s_1)x_3 - a\theta(s_2)x_1 - b\theta(s_3)x_1\big) \\
&= a([s_1, s_2], [x_1, x_2] + \theta(s_1)x_2 - \theta(s_2)x_1) \\
&\qquad + b([s_1, s_3], [x_1, x_3] + \theta(s_1)x_3 - \theta(s_3)x_1) \\
&= a[(s_1, x_1), (s_2, x_2)] + b[(s_1, x_1), (s_3, x_3)]
\end{aligned}$$

Thus the bracket is linear in the second entry. Now we show that the bracket of something with itself is zero.

$$[(s_1, x_1), (s_1, x_1)] = ([s_1, s_1], [x_1, x_1] + \theta(s_1)x_1 - \theta(s_1)x_1) = (0, 0)$$

I am too lazy to prove that the Jacobi identity holds for this Lie algebra, because it's very technical and boring. □

Proposition 8.17 (Exercise 3.9ii)**.** *In the construction in the above proposition, the bracket on $S \oplus I$ is a semidirect product of I by S.*

Proof. To show: $\{0\} \oplus I$ is an ideal of $S \oplus I$ and $S \oplus \{0\}$ is a subalgebra of $S \oplus I$. Let $(s_1, 0), (s_2, 0) \in S \oplus \{0\}$. Then

$$[(s_1, 0), (s_2, 0)] = ([s_1, s_2], [0, 0] + \theta(s_1)0 - \theta(s_2)0) = ([s_1, s_2], 0)$$

Thus $S \oplus \{0\}$ is a subalgebra of $S \oplus I$. Let $(s_1, x_1) \in S \oplus I$, and $(0, x_2) \in I$. Then

$$[(s_1, x_1), (0, x_2)] = ([s_1, 0], [x_1, x_2] + \theta(s_1)x_2 - 0x_1) = (0, [x_1, x_2] + \theta(s_1)x_2)$$

Thus $I \oplus \{0\}$ is an ideal of $S \oplus I$. □

9 Chapter 4 Exercises

Proposition 9.1 (Exercise on page 31, section 4.2)**.** *Let L be a Lie algebra. Then $L^{(k)} \subseteq L^k$. As a consequence, every nilpotent algebra is solvable.*

Proof. This is true for $n = 1$ since $L^{(1)} = L^1 = L'$. Suppose that $L^{(n)} \subseteq L^n$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned} L^{(n+1)} &= [L^{(n)}, L^{(n)}] = \operatorname{span}\{[x, y] : x, y \in L^{(n)}\} \subseteq \operatorname{span}\{[x, y] : x, y \in L^n\} \\ L^{n+1} &= [L, L^n] = \operatorname{span}\{[x, y] : x \in L, y \in L^n\} \end{aligned}$$

Since $L^n \subseteq L$,

$$L^{(n+1)} \subseteq \operatorname{span}\{[x, y] : x, y \in L^n\} \subseteq \operatorname{span}\{[x, y] : x \in L, y \in L^n\} \subseteq L^{n+1}$$

Thus by induction, $L^{(k)} \subseteq L^k$ for all $k \in \mathbb{N}$. This implies that every nilpotent algebra is solvable, because if $L^k = 0$, then $L^{(k)} \subseteq L^k = 0$ so $L^{(k)} = 0$. □

Proposition 9.2 (Exercise 4.1)**.** *Let $\phi : L_1 \to L_2$ be an onto homomorphism. Then $\phi(L_1^{(k)}) = L_2^{(k)}$.*

Proof. The statement is true for $k = 1$ as proved in Exercise 2.8a. Suppose the statement is true for $k = n$. We will show that this implies that it is true for $k = n + 1$.

$$\begin{aligned} \phi(L_1^{(n+1)} &= \phi([L_1^{(n)}, L_1^{(n)}]) \\ &= \phi(\operatorname{span}\{[x, y] : x, y \in L_1^{(n)}\} \\ &= \operatorname{span}\{\phi([x, y]) : x, y \in L_1^{(n)}\} \\ &= \operatorname{span}\{[\phi(x), \phi(y)] : x, y \in L_1^{(n)}\} \\ &= \operatorname{span}\{[w, z] : w, z \in L_2^{(n)}\} && \text{since } \phi \text{ is onto} \\ &= [L_2^{(n)}, L_2^{(n)}] \\ &= L_2^{(n+1)} \end{aligned}$$

Thus by induction the statement is true for all $k \in \mathbb{N}$. □

Definition 9.3 (definition for Exercise 4.2)**.** $\operatorname{sp}(2k, \mathbb{C}) = \operatorname{gl}_S(2k, \mathbb{C})$ *where S is the matrix*

$$\begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$$

Proposition 9.4 (Exercise 4.2)**.** *For $x \in \operatorname{gl}(2k, \mathbb{C})$, $x \in \operatorname{sp}(2k, \mathbb{C})$ if and only if x is of the form*

$$\begin{pmatrix} m & p \\ q & -m^t \end{pmatrix}$$

for square $k \times k$ matrices p, q, m where p, q are symmetric.

Proof. Suppose that $x \in \mathrm{gl}(2k, \mathbb{C})$ is of the proposed form. Then

$$\begin{aligned} x^t S &= \begin{pmatrix} m^t & q \\ p & -m \end{pmatrix} \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix} = \begin{pmatrix} -q & m^t \\ m & p \end{pmatrix} \\ -Sx &= \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix} \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix} = \begin{pmatrix} -q & m^t \\ m & p \end{pmatrix} \end{aligned}$$

Thus $x^t S = -Sx$. Now suppose that $x \in^(2k, \mathbb{C})$. Then

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some $a, b, c, d \in \mathrm{gl}(k, \mathbb{C})$. We know that $x^t S = -Sx$, so

$$\begin{aligned} \begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix} \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix} &= \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \begin{pmatrix} -c^t & a^t \\ -d^t & b^t \end{pmatrix} &= \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \end{aligned}$$

Thus $c = c^t, b = b^t$, and $d = -a^t$. Thus x is of the desired form. ☐

Proposition 9.5 (Exercise 4.3)**.** *Let L be a solvable Lie algebra. Then* $\mathrm{ad}\, L$ *is a solvable subalgebra of* $\mathrm{gl}(L)$*.*

Proof. We know that $\mathrm{ad} : L \to \mathrm{gl}(L)$ is a homomorphism, so $\mathrm{ad}\, L$ is a subalgebra of $\mathrm{gl}(L)$. By Lemma 4.4a, every homomorphic image of L is solvable, so $\mathrm{ad}\, L$ is solvable. ☐

Proposition 9.6 (Exercise 4.3)**.** *If L is Lie algebra such that* $\mathrm{ad}\, L$ *is a solvable Lie subalgebra of* $\mathrm{gl}(L)$*, then L is solvable.*

Proof. We know that $\ker \mathrm{ad} = Z(L)$, by the First Isomorphism Theorem, $L/Z(L) \cong \mathrm{ad}\, L$. Since $Z(L)' = 0$, $Z(L)$ is solvable, and by hypothesis $\mathrm{ad}\, L$ is solvable, so $L/Z(L)$ is solvable. Then $Z(L)$ is an ideal of L with $Z(L)$ and $L/Z(L)$ solvable, so by Lemma 4.4b, L is solvable. ☐

Lemma 9.7 (for Exercise 4.3)**.** *Let $\phi : L_1 \to L_2$ be an onto homomorphism. Then $\phi(L^k) = L_2^k$.*

Proof. We know that $\phi(L_1) = L_2$ and $\phi(L_1') = L_2'$ by Exercise 2.8a. Suppose that $\phi(L_1^n) = L_2^n$ for some n. Then

$$\begin{aligned} \phi(L_1^{n+1}) &= \phi(\mathrm{span}\{[x, y] : x \in L_1, y \in L_1^n\}) \\ &= \mathrm{span}\{\phi([x, y]) : x \in L_1, y \in L_1^n\} \\ &= \mathrm{span}\{[\phi(x), \phi(y)] : x \in L_1, y \in L_1^n\} \\ &= \mathrm{span}\{[w, z] : L \in L_2, z \in L_2^n\} \\ &= L_2^{n+1} \end{aligned}$$

Thus by induction, $\phi(L_1^k) = L_2^k$ for all $k \in \mathbb{N}$. ☐

Lemma 9.8 (for Exercise 4.3)**.** *Let L be a nilpotent Lie algebra, and let $\phi : L \to M$ be a homomorphism. Then $\phi(L)$ is a nilpotent subalgebra of M.*

Proof. We know from Exercise 1.6 that $\phi(L)$ is a subalgebra of M. Since L is nilpotent, $L^k = 0$ for some k. Then by the previous lemma, $\phi(L^k) = \phi(L)^k = 0$, so $\phi(L)$ is nilpotent. □

Proposition 9.9 (Exercise 4.3)**.** *If L is nilpotent, then $\operatorname{ad} L$ is a nilpotent subalgebra of $\operatorname{gl}(L)$.*

Proof. Let $\pi : L \to L/Z(L)$ be defined by $\pi(x) = x + Z(L)$. This is an onto homomorphism by Exercise 2.3ii. Thus since L is nilpotent, $\phi(L) = L/Z(L)$ is nilpotent. By the 1st Isomorphism Theorem, $L/\ker \operatorname{ad} = L/Z(L) \cong \operatorname{ad} L$ so $\operatorname{ad} L$ is nilpotent. □

Proposition 9.10 (Exercise 4.3)**.** *If $\operatorname{ad} L$ is a nilpotent subalgebra of $\operatorname{gl}(L)$, then L is nilpotent.*

Proof. We know that $\operatorname{ad} : L \to \operatorname{gl}(L)$ is a homomorphism, with $\ker \operatorname{ad} = Z(L)$. By the 1st Isomorphism Theorem, $L/\ker \operatorname{ad} = L/Z(L) \cong \operatorname{ad} L$. Thus $L/Z(L)$ is nilpotent, so by Lemma 4.9b, L is nilpotent. □

Proposition 9.11 (Exercise 4.4)**.** *Let $L = \operatorname{n}(n, F)$. Then L^k has a basis consisting of e_{ij} where $i < j - k$. Thus L is nilpotent. Furthermore, the smallest k such that $L^k = 0$ is $k = n$.*

Proof. This is true for $k = 0$ by definition of $\operatorname{n}(n, F)$. Suppose it is true for some $k \geq 0$. Then

$$\begin{aligned}
L^{k+1} &= [L, L^k] \\
&= \operatorname{span}\{[x, y] : x \in L, y \in L^k\} \\
&= \operatorname{span}\{[e_{ij}, e_{ab}] : i < j, a + k < b\} \\
&= \operatorname{span}\{\delta_{ja} e_{ib} - \delta_{ib} e_{aj} : i < j, a + k < b\}
\end{aligned}$$

$$\delta_{ja} e_{ib} - \delta_{ib} e_{aj} = \begin{cases} 0 & j \neq a, i \neq b \\ e_{il} & j = a, i \neq b \\ -e_{kj} & j \neq a, i = b \\ e_{ii} - e_{aa} & j = a, i \neq b \end{cases}$$

The fourth possibility never happens since we know that $i < j$. The first case contributes nothing to the span. In the second case, we have e_{ib} where $i < j = a < b - k$ so $i < b - k$. Likewise in the third case, we have $-e_{aj}$ where $a + k < b = j$ so $a < j - k$. Thus

$$L^{k+1} = \operatorname{span}\{e_{ij} : i < j - k\}$$

So by induction this is true for all $k \in \mathbb{N}$.

Now we show that $\operatorname{n}(n, F)$ is nilpotent. Let $k = n$. Then L^k has a basis e_{ij} where $i < j - n$, but this is an empty set, since $j \leq n$ and $i \geq 1$. Thus $L^k = 0$. To see that n is the smallest such k, suppose that $k < n$. Then L^k has a basis of e_{ij} with $i < j - k$ and if $k < n$ this is not empty. □

Proposition 9.12 (Exercise 4.5i)**.** $\mathrm{b}(n, F)' = \mathrm{n}(n, F)$

Proof.

$$\begin{aligned} \mathrm{b}(n, F) &= \operatorname{span}\{e_{ab} : a \leq b\} \\ \mathrm{b}(n, F)' &= \operatorname{span}\{[e_{ij}, e_{kl}] : i \leq j, k \leq l\} \\ &= \operatorname{span}\{\delta_{jk} e_{il} - \delta_{il} e_{kj} : i \leq j, k \leq l\} \end{aligned}$$

$$\delta_{jk} e_{il} - \delta_{il} e_{kj} = \begin{cases} 0 & j \neq k, i \neq l \\ e_{il} & j = k, i \neq l \\ -e_{kj} & j \neq k, i = l \\ e_{ii} - e_{kk} & j = k, i \neq l \end{cases}$$

In the fourth case, $i \leq j = k \leq l = i$, so $i = j = k = l$, so this bracket product turns out to be $e_{ii} - e_{ii} = 0$, so it contributes nothing to the span.

In the second case, we get e_{il} where $i \leq j = k \leq l$ and $i \neq l$, so $i < l$. In the third case, we have $-e_{kj}$ where $k \leq l = i \leq j$ and $k \neq j$ so $k < j$. Thus all of the bracket products in $\mathrm{b}(n, F)$ are of the form $\pm e_{ij}$ where $i < j$. Thus

$$\mathrm{b}(n, F) = \operatorname{span}\{e_{ij} : i < j\} = \mathrm{n}(n, F)$$

□

Proposition 9.13 (Exercise 4.5ii)**.** *Let* $L = \mathrm{b}(n, F)$*. Then as basis for* $L^{(m)}$ *is*

$$\{e_{ij} : i \leq j - 2^{k-1}\}$$

Proof. For $k = 1$ this is true by 4.5i. Suppose that it is true for some $m \geq 1$. Then $L^{(m)} = \operatorname{span}\{e_{ij} : i \leq j - 2^{k-1}\}$. Then

$$\begin{aligned} L^{(m+1)} &= [L^{(m)}, L^{(m)}] \\ &= \operatorname{span}\{[e_{ij}, e_{kl}] : i \leq j - 2^{m-1}, k \leq l - 2^{m-1}\} \\ &= \operatorname{span}\{\delta_{jk} e_{il} - \delta_{il} e_{kj} : i \leq j - 2^{m-1}, k \leq l - 2^{m-1}\} \end{aligned}$$

$$\delta_{jk} e_{il} - \delta_{il} e_{kj} = \begin{cases} 0 & j \neq k, i \neq l \\ e_{il} & j = k, i \neq l \\ -e_{kj} & j \neq k, i = l \\ e_{ii} - e_{kk} & j = k, i \neq l \end{cases}$$

Since $m \geq 1$, $2^{m-1} \geq 0$, so $i \leq j - 2^{m-1}$ implies that $i < j$. This rules out the fourth case, since in the fourth case $i = j = k = l$. Thus in both the first and fourth cases, the the brackets do not contribute to the span. In the second case,

$$\begin{aligned} i \leq j - 2^{m-1} &\implies i + 2^{m-1} \leq j = k \leq l - 2^{m-1} \\ &\implies i + 2^m \leq l \\ &\implies i \leq l - 2^m \end{aligned}$$

And in the third case,

$$\begin{aligned}k \leq l - 2^{m-1} &\implies k + 2^{m-1} \leq l = i \leq j - 2^{m-1} \\ &\implies k \leq j - 2^m\end{aligned}$$

Thus $L^{(m+1)}$ is spanned by matrices of the form e_{ij} where $i \leq j - 2^{m-1}$. Thus

$$L^{(m+1)} = \operatorname{span}\{e_{ij} : i \leq j - 2^{m-1}\}$$

So by induction, the proposition is true for all $k \in \mathbb{N}$. ☐

Proposition 9.14 (Exercise 4.5iii). *$L = \mathrm{b}(n, F)$ is solvable, and the smallets k such that $L^{(k)} = 0$ is the smallest integer k satisfying $k > \log_2(n-1) + 1$.*

Proof. First we show that if $k > \log_2(n-1) + 1$, then $L^{(k)} = 0$. If k satisfies this inequality, then

$$k - 1 > \log_2(n-1) \implies 2^{k-1} > n - 1 \implies 1 + 2^{k-1} > n$$

Let e_{ij} be a basis element of $L^{(k)}$. Then $1 \leq i \leq j - 2^{k-1}$, which implies $1 + 2^{k-1} \leq n$. However, the above just showed that $1 + 2^{k-1} > n$, so there can be no i, j satisfying $1 \leq i \leq j - 2^{k-1} \leq n$. Thus there are no basis elements of $L^{(k)}$, so $L^{(k)} = 0$, so L is solvable.

Now suppose that $k \leq \log_2(n-1) + 1$. Then

$$\begin{aligned}k - 1 \leq \log_2(n-1) &\implies 2^{k-1} \leq n - 1 \\ &\implies 1 + 2^{k-1} \leq n\end{aligned}$$

So if $i = 1$ and $j = 1 + 2^{k-1}$, then

$$1 + 2^{k-1} \leq i + 2^{k-1} \leq j \leq n + 2^{k-1} \implies 1 \leq i \leq j - 2^{k-1} \leq n$$

So if $k \leq \log_2(n-1) + 1$, then $L^{(k)}$ has a non-empty basis containing e_{ij} where $i = 1, j = 1 + 2^{k-1}$. Thus the proposed value for k is a minimum to get $L^{(k)} = 0$. ☐

Proposition 9.15 (Exercise 4.6). *Let L be a semisimple Lie algebra. Then L has no non-zero abelian ideals.*

Proof. Let I be an abelian ideal of L. Then $I' = 0$, so I is solvable. Since L has no non-zero solvable ideals, $I = 0$. Thus all abelian ideals of L are the zero ideal, so L has no non-zero abelian ideals. ☐

Proposition 9.16 (Exercise 4.5iv). *If $n \geq 2$, then $L = \mathrm{b}(n, F)$ is not nilpotent.*

Proof. We know that $L' = \mathrm{n}(n, F)$. We compute $L^{(2)}$:

$$\begin{aligned}L^{(2)} &= [L, L'] \\ &= \operatorname{span}\{[e_{ij}, e_{kl}] : i \leq j, k < l\} \\ &= \operatorname{span}\{\delta_{jk} e_{il} - \delta_{il} e_{kj} : i \leq j, k < l\}\end{aligned}$$

$$\delta_{jk} e_{il} - \delta_{il} e_{kj} = \begin{cases} 0 & j \neq k, i \neq l \\ e_{il} & j = k, i \neq l \\ -e_{kj} & j \neq k, i = l \\ e_{ii} - e_{kk} & j = k, i \neq l \end{cases}$$

In the first cases, nothing is contributed to the span. The fourth case can never happen, since it would imply $j = k < l = i \leq j$ which means $j < j$, an impossibility. In the second case, we have e_{il} where $i \leq j = k < l$ so $i < l$, and in the third case we have $-e_{kj}$ where $k < l = i \leq j$ so $k < j$.

Thus $L^{(2)}$ is the span of matrices e_{ij} with $i < j$, which is $\mathrm{n}(n, F)$. Thus $L^{(k)} = \mathrm{n}(n, F)$ for all $k \in \mathbb{N}$, so L is not nilpotent. □

Proposition 9.17 (Exercise 4.6)**.** *Let L be a Lie algebra with no non-zero abelian ideals. Then L is semisimple.*

Proof. Let I be a solvable ideal of L. Then $I^{(k)} = 0$ for some k. Let m be the minimum of all such k, so $I^{(m)} = 0$ but $I^{(m-1)} \neq 0$. Then $I^{(m-1)}$ is an abelian ideal of L, so $I^{(m-1)} = 0$. So we have a contradiction, that $I^{(m-1)} = 0$ and $I^{(m-1)} \neq 0$. Thus we conclude that L has no solvable ideals. □

Lemma 9.18 (Exercise 4.7)**.** *Let $I \subset \mathrm{sl}(n, \mathbb{C})$ be an ideal with $e_{ij} \in I$ for some $i \neq j$. Then $e_{ii} - e_{jj} \in I$.*

Proof. Since I is an ideal, $[e_{ij}, e_{ji}] = \delta_{jj} e_{ii} - \delta_{ii} e_{jj} = e_{ii} - e_{jj} \in I$. □

Lemma 9.19 (Exercise 4.7)**.** *Let $I \subset \mathrm{sl}(n, \mathbb{C})$ be an ideal with $e_{ii} - e_{jj} \in I$ for some $i \neq j$. Then $e_{im}, e_{mi} \in I$ for all $m \neq i$ and $e_{jm}, e_{mj} \in I$ for all $m \neq j$.*

Proof. First we compute the bracket of h with some general e_{kl} (with $k \neq l$) which we know is in I since I is an ideal.

$$\begin{aligned}
[h, e_{kl}] &= [e_{ii}, e_{kl}] - [e_{i+1,i+1}, e_{kl}] \\
&= (\delta_{ik} e_{il} - \delta_{il} e_{ki}) - (\delta_{jk} e_{jl} - \delta_{jl} e_{kj}) \\
&= \delta_{ik} e_{kl} - \delta_{il} e_{kl} - \delta_{jk} e_{kl} + \delta_{jl} e_{kl} \\
&= (\delta_{ik} - \delta_{jk} + \delta_{jl} - \delta_{il}) e_{kl}
\end{aligned}$$

Now we need to enumerate the cases for this coefficient involving several Kronecker deltas.

$$\delta_{ik} - \delta_{jk} = \begin{cases} 1 & i = k \\ 0 & i \neq k, j \neq k \\ -1 & j = k \end{cases} \qquad \delta_{jl} - \delta_{il} = \begin{cases} 1 & j = l \\ 0 & j \neq l, i \neq l \\ -1 & i = l \end{cases}$$

$$\delta_{ik} - \delta_{jk} + \delta_{jl} - \delta_{il} = \begin{cases} 2 & i = k, j = l \\ 1 & i = k, j \neq l \text{ OR } j = l, i \neq k \\ 0 & i \neq k, i \neq l, j \neq k, j \neq l \\ -1 & j = k, i \neq l \text{ OR } i = l, j \neq k \\ -2 & i = l, j = k \end{cases}$$

So we see that the only time that this coefficient is zero is when $i \neq k, i \neq l, j \neq k, j \neq l$. Now suppose $I \subset \mathrm{sl}(n, \mathbb{C})$ is an ideal containing $h = e_{ii} - e_{jj}$, and $m \neq i$. By the previous computation,

$$\begin{aligned}
[h, e_{mi}] &= \lambda_1 e_{mi} \\
[h, e_{im}] &= \lambda_2 e_{im}
\end{aligned}$$

where $\lambda_1, \lambda_2 \in \{-2, -1, 1, 2\}$, and thus $e_{mi}, e_{im} \in I$. Likewise for $m \neq j$, we have

$$[h, e_{mj}] = \lambda_3 e_{mj}$$
$$[h, e_{jm}] = \lambda_3 e_{jm}$$

where $\lambda_3, \lambda_4 \in \{-2, -1, 1, 2\}$, and thus $e_{mj}, e_{jm} \in I$. □

Lemma 9.20 (Exercise 4.7)**.** *Let $I \subset \mathrm{sl}(n, \mathbb{C})$ be an ideal such that $e_{ii} - e_{jj} \in I$. Then $e_{ll} - e_{kk} \in I$ for all $1 \leq k, l \leq n$ with $k \neq l$.*

Proof. We have $e_{ii} - e_{jj} \in I$. By Lemma 9.19, $e_{ik}, e_{li} \in I$ for each $k, l \neq i$. Then by Lemma 9.18, $e_{ii} - e_{kk}, e_{ll} - e_{ii} \in I$ for each $k, l \neq i$. Then since I is a vector subspace,

$$(e_{ii} - e_{kk}) + (e_{ll} - e_{kk}) = e_{ll} - e_{kk} \in I$$

This gives us each $e_{ll} - e_{kk}$ where $l, k \neq i$. The same process may be used to generate each $e_{ll} - e_{kk}$ where $l, k \neq j$. Note that along the way, we genereated each $e_{ii} - e_{kk}$ with $i \neq k$, and finally note that we began by having $e_{ii} - e_{jj}$, which covers the case where $i = l, j = k$. □

Lemma 9.21 (Exercise 4.7)**.** *Let $I \subset \mathrm{sl}(n, \mathbb{C})$ be an ideal such that $e_{ij} \in I$ or $e_{ii} - e_{jj} \in I$ for some $i \neq j$. Then $I = \mathrm{sl}(n, \mathbb{C})$.*

Proof. If $e_{ij} \in I$, then $e_{ii} - e_{jj} \in I$, so either way we may assume $e_{ii} - e_{jj} \in I$ for some $i \neq j$. Then by Lemma 9.20, I contains all diagonal elements of $\mathrm{sl}(n, \mathbb{C})$. Let $k \neq l$ with $1 \leq k, l \leq n$. Then $e_{kk} - e_{ll} \in I$, so by Lemma 9.19, $e_{kl} \in I$. Thus I contains the standard basis for $\mathrm{sl}(n, \mathbb{C})$, so $I = \mathrm{sl}(n, \mathbb{C})$. □

The above lemma gives the significance of the previous three lemmas. This lemma says that any nonzero ideal of $\mathrm{sl}(n, \mathbb{C})$ containing just one of the usual basis elements is the entirety of $\mathrm{sl}(n, \mathbb{C})$. Now we just need to show that any nonzero ideal of $\mathrm{sl}(n, \mathbb{C})$ contains one of these ususal basis elements.

Lemma 9.22 (Exercise 4.7)**.** *Let $v \in \mathrm{sl}(n, \mathbb{C})$ where*

$$v = \sum_{i \neq j} c^{ij} e_{ij} + \sum_{i=1}^{n} d^i e_{ii}$$

and let $k \neq l$. Then

$$[e_{kl}, [e_{kl}, v]] = -2c^{lk} e_{kl}$$

Proof. We can write v as $v = d + n$ where d is a diagonal matrix and n is a matrix with zeros on the diagonal. Then by linearity of the bracket,

$$[e_{kl}, [e_{kl}, v]] = [e_{kl}, [e_{kl}, d] + [e_{kl}, n]] = [e_{kl}, [e_{kl}, d]] + [e_{kl}, [e_{kl}, n]]$$

We claim that $[e_{kl}, [e_{kl}, d]] = 0$. By a previous lemma, since d is diagonal, $[e_{kl}, d] = \lambda e_{kl}$, so then we have a bracket of e_{kl} with a multiple of e_{kl}, which will be zero. So $[e_{kl}, [e_{kl}, v]] =$

$[e_{kl}, [e_{kl}, n]]$.

$$\begin{aligned}
[e_{kl}, [e_{kl}, v]] &= \sum_{i \neq j} c^{ij} [e_{kl}, [e_{kl}, e_{ij}]] \\
&= \sum_{i \neq j} c^{ij} [e_{kl}, \delta_{il} e_{kj} - \delta_{kj} e_{il}] \\
&= \sum_{i \neq j} c^{ij} (\delta_{il} [e_{kl}, e_{kj}] - \delta_{kj} [e_{kl}, e_{il}]) \\
&= \sum_{i \neq j} c^{ij} (\delta_{il} (\delta_{lk} e_{kj} - \delta_{kj} e_{kl}) - \delta_{kj} (\delta_{il} e_{kl} - \delta_{kl} e_{il})) \\
&= \sum_{i \neq j} c^{ij} (\delta_{il} \delta_{lk} e_{kj} + \delta_{il} \delta_{kj} e_{kl} - \delta_{kj} \delta_{il} e_{kl} + \delta_{kj} \delta k l e_{il}) \\
&= \sum_{i \neq j} c^{ij} (\delta_{kli} e_{ij} + \delta_{klj} e_{ij} - 2 \delta_{il} \delta_{kj} e_{kl})
\end{aligned}$$

At this point, note that since $k \neq l$, we have $\delta_{kli} = \delta_{klj} = 0$, so we can cross out the e_{ij} terms.

$$[e_{kl}, [e_{kl}, v]] = \sum_{i \neq j} c^{ij} (-2 \delta_{il} \delta_{kj} e_{kl}) = (-2) \sum_{i \neq j} c^{ij} \delta_{il} \delta k j e_{kl}$$

The only nonzero term of this summation occurs when $i = l$ and $k = j$, so

$$[e_{kl}, [e_{kl}, v]] = -2 c^{lk} e_{kl}$$

□

Lemma 9.23 (Exercise 4.7)**.** *Let $v \in \mathrm{sl}(n, \mathbb{C})$ be a diagonal matrix. We can write v as $\sum_{i=1}^n d^i e_{ii}$ (where $\sum_i d^i = 0$). Then for $k \neq l$,*

$$[v, e_{kl}] = (d^k - d^l) e_{kl}$$

Proof.

$$[v, e_{kl}] = \sum_{i=1}^n d^i [e_{ii}, e_{kl}] = \sum_{i=1}^n d^i (\delta_{ik} e_{il} - \delta_{il} e_{ki}) = e_{kl} \sum_{i=1}^n d^i (\delta_{ik} - \delta i l) = e_{kl} (d^k - d^l)$$

□

Lemma 9.24 (Exercise 4.7)**.** *Let $v \in \mathrm{sl}(n, \mathbb{C})$ be a nonzero diagonal matrix. Then there exist k, l with $k \neq l$ such that $[v, e_{kl}] = \lambda e_{kl}$ for some $\lambda \neq 0$ ($\lambda \in \mathbb{C}$.)*

Proof. Suppose $v = \sum_{i=1}^n d^i e_{ii}$, where $\sum_{i=1}^n d_i = 0$ and some $d^i \neq 0$. Then for $k \neq l$, by previous lemma, $[v, e_{kl}] = (d^k - d^l) e_{kl}$. Suppose to the contrary that $[v, e_{kl}] = 0$ for all k, l. Then $d^k - d^l = 0 \implies d^k = d^l$ for all k, l. But then since $\sum_{i=1}^n d_i = 0$, this implies that $d_i = 0$ for all i. But v is nonzero by hypothesis, so we conclude that for some d^k, d^l, we have $d^k - d^l \neq 0$, so we reach our desired conclusion. □

Recall that 9.21 tells us that any ideal of $\mathrm{sl}(n,\mathbb{C})$ that contains just one of the usual basis elements contains all of $\mathrm{sl}(n,\mathbb{C})$. Lemmas 9.22, 9.23, 9.24 allow us to show that any nonzero ideal of $\mathrm{sl}(n,\mathbb{C})$ contains one of the usual basis elements.

Lemma 9.25 (Exercise 4.7)**.** *Let $I \subset \mathrm{sl}(n,\mathbb{C})$ be a nonzero ideal. Then $e_{ij} \in L$ for some $i \neq j$.*

Proof. Let $v \in L$ be nonzero, and write v as

$$v = \sum_{i \neq j} c^{ij} e_{ij} + \sum_{i=1}^{n} d^i e_{ii}$$

Suppose that v is not diagonal, that is, that some $c^{ij} \neq 0$. Then by 9.22,

$$[e_{ji}, [e_{ji}, v]] = -2c^{ij} e_{ji} \in I$$

and since $c^{ij} \neq 0$, we have $e_{ji} \in I$. Now suppose that v is diagonal. Then by 9.24, there exists e_{kl} such that

$$[e_{kl}, v] = \lambda e_{kl}$$

where $\lambda \neq 0$. Then $e_{kl} \in I$. $\square$

Theorem 9.26 (Exercise 4.7)**.** $\mathrm{sl}(n,\mathbb{C})$ *is a simple Lie algebra for $n \geq 2$.*

Proof. Let I be a nonzero ideal of $\mathrm{sl}(n,\mathbb{C})$. By 9.25, there exists $e_{ij} \in I$ with $i \neq j$. Then by 9.21, $I = \mathrm{sl}(n,\mathbb{C})$. Thus $\mathrm{sl}(n,\mathbb{C})$ has no nonzero proper ideals. $\square$

Definition 9.27. *Let $A \in \mathrm{gl}(n,F)$. Then A_{ij} is the $(n-1) \times (n-1)$ matrix formed by deleting the ith row and jth column of A.*

Proposition 9.28 (Exercise 4.9i)**.** *Let $A \in \mathrm{gl}(n,F)$ and let I_n be the identity matrix for $\mathrm{gl}(n,F)$, and let $\lambda \in F$. Then $\det(I_n + \lambda A)$ is a polynomial in λ with constant term 1 and linear term $\lambda(\mathrm{tr}\, A)$.*

Proof. We show directly that this is true for $n = 2$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then

$$\begin{aligned} \det(I + \lambda A) &= \det \begin{pmatrix} 1 + \lambda a & \lambda b \\ \lambda c & 1 + \lambda d \end{pmatrix} \\ &= (1 + \lambda a)(1 + \lambda d) + \lambda^2 bc \\ &= 1 + \lambda(a + d) + \lambda^2(ad + bc) \\ &= 1 + \lambda(\mathrm{tr}\, A) + \lambda^2(ad + bc) \end{aligned}$$

Now we prove the general case by induction. Suppose that for every $A \in \mathrm{gl}(k, F)$, we know that $\det(I + \lambda A) = 1 + \lambda(\operatorname{tr} A) + \lambda^2 P(\lambda)$ where P is some irrelevant polynomial in λ. Let $B \in \mathrm{gl}(k+1, F)$ where $B = (b_{ij})$. Then

$$I + \lambda B = \begin{pmatrix} 1 + \lambda b_{11} & \lambda b_{12} & \dots \\ \lambda b_{21} & 1 + \lambda b_{22} & \dots \\ \vdots & \vdots & \end{pmatrix}$$

By Laplacian expansion along the first row,

$$\det(I + \lambda B) = (1 + \lambda b_{11}) \det(I_{11} + \lambda B_{11}) + \sum_{i=2}^{k+1} \lambda b_{1i} \det(I_{1i} + \lambda B_{1i})$$

We claim that the summation term contributes nothing to the constant or linear terms of this polynomial in λ. Note that for $i \neq 1$, I_{1i} will always have a zero row, so $I_{1i} + \lambda B_{1i}$ has a row where every entry is divisible by λ. One could compute the determinant of $I_{1i} + \lambda B_{1i}$ by Laplacian expansion along this row, and every term in the sum would be divisible by λ, so we can conclude that for every $2 \leq i \leq k+1$, $\lambda | \det(I_{1i} + \lambda B_{1i}$. Thus each term in the sum

$$\sum_{i=2}^{k+1} \lambda b_{1i} \det(I_{1i} + \lambda B_{1i})$$

is divisible by λ^2. Thus it contributes nothing to the constant or linear term. Finally, utilizing our inductive hypothesis,

$$\det(I_{11} + \lambda B_{11}) = 1 + \lambda \operatorname{tr} B_{11} + \lambda^2 P_1(\lambda)$$

Thus

$$\begin{aligned} \det(I + \lambda B) &= (1 + \lambda b_{11})\big(\det(I_{11} + \lambda B_{11})\big) + \lambda^2 P_2(\lambda) \\ &= (1 + \lambda b_{11})\big(1 + \lambda \operatorname{tr} B_{11} + \lambda^2 P_1(\lambda)\big) + \lambda^2 P_2(\lambda) \\ &= 1 + \lambda \operatorname{tr} B_{11} + \lambda^2 P_2(\lambda) + \lambda b_{11} + \lambda^2 b_{11} \operatorname{tr} B_{11} + \lambda^3 P_2(\lambda) + \lambda^2 P_1(\lambda) \\ &= 1 + \lambda(\operatorname{tr} B_{11} + b_{11}) + \lambda^2 P_3(\lambda) \\ &= 1 + \lambda \operatorname{tr} B + \lambda^2 P_3(\lambda) \end{aligned}$$

Where P_1, P_2, P_3 are are all irrelevant polynomials in λ. This completes the proof by induction. $\square$

Proposition 9.29 (Exercise 4.9iia)**.** *Let $S \in \mathrm{gl}(n, \mathbb{C})$ and let $(,) : \mathbb{C}^n \to \mathbb{C}^n$ be the complex bilinear form with matrix S. Let*

$$G_S(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) : (Av, Av) = (v, v) \textit{ for } v \in \mathbb{C}^n\}$$

We claim that $G_S(n, \mathbb{C})$ is a group under usual matrix multiplication.

Proof. Clearly $I_n \in G_S(n, \mathbb{C})$, because $(I_n v, I_n v) = (v, v)$. Thus $G_S(n, \mathbb{C})$ has an identity. Associativity of matrix multiplication is inherited from $GL(n, \mathbb{C})$. Now we show that $GL(n, \mathbb{C})$ is contains inverses. Let $A \in G_S(n, \mathbb{C})$. Since $A \in GL(n, \mathbb{C})$, A^{-1} exists. Also, $(v, v) = (Av, Av)$, so

$$(A^{-1}v, A^{-1}v) = A^{-2}(v, v) = A^{-2}(Av, Av) = A^{-2}A^2(v, v) = (v, v)$$

Thus $A^{-1} \in G_S(n, \mathbb{C})$. Now we show that $G_S(n, \mathbb{C})$ is closed under matrix multiplication. Let $A, B \in G_S(n, \mathbb{C})$. Then $(v, v) = (Av, Av) = (Bv, Bv)$. Then

$$(ABv, ABv) = A^2(Bv, Bv) = A^2(v, v) = (Av, Av) = (v, v)$$

Thus $AB \in G_S(n, \mathbb{C})$. $\square$

Proposition 9.30 (Exercise 4.9iiia)**.** *Let* $G = \{A \in \mathrm{gl}(n, \mathbb{C}) : A^t = A^{-1}\}$. *Then* G *is a group under matrix multiplication.*

Proof. Clearly $I_n \in G$ since $I_n^t = I_n = I_n^{-1}$. Associativity is inherited from $\mathrm{gl}(n, \mathbb{C})$. For $A \in G$, we have $(A^{-1})^t = (A^t)^{-1}$ so $A^{-1} \in G$. For $A, B \in G$, $(AB)^t = B^t A^t = B^{-1}A^{-1} = (AB)^{-1}$ so $AB \in G$. $\square$

10 Chapter 5 Exercises

Proposition 10.1 (Exercise 5.1i)**.** *Let V be a vector space over F, and let $A \subseteq \mathrm{gl}(V)$ be a subalgebra. Let $\lambda : A \to F$ be a linear map. Let*

$$V_\lambda = \{v \in V : a(v) = \lambda(a)v \textit{ for all } a \in A\}$$

Then V_λ is a vector subspace of V.

Proof. V_λ contains the zero vector since $a : V \to V$ and $\lambda : A \to F$ are linear maps, so $a(0) = 0, \lambda(0) = 0 \implies a(0) = \lambda(0)v = 0$. Now let $v, w \in V_\lambda$. Then $a(v) = \lambda(a)v$ and $a(w) = \lambda(a)w$ for all $a \in A$. Then by linearity of a and λ,

$$\begin{aligned} a(v) + a(w) &= \lambda(a)v + \lambda(a)w \\ a(v + w) &= \lambda(a)(v + w) \end{aligned}$$

Thus $v + w \in V_\lambda$. Let $v \in V_\lambda, b \in F$. Then

$$a(v) = \lambda(a)v \implies ba(v) = b\lambda(a)v \implies a(bv) = \lambda(a)(bv)$$

Thus $bv \in V_\lambda$. Thus V_λ is closed under vector addition and scalar multiplication, so it is a vector subspace of V. $\square$

Proposition 10.2 (Exercise 5.1ii)**.** *Let $A = \mathrm{d}(n, F) \subseteq \mathrm{gl}(n, F)$ and let $V = F^n$. Let $\{e_1, \ldots e_n\}$ be the standard basis for V. For $a \in A$, denote the entries by a^i, that is,*

$$a = \begin{pmatrix} a^1 & 0 & \ldots \\ 0 & a^2 & \ldots \\ \vdots & \vdots & \\ 0 & \ldots & a^n \end{pmatrix}$$

Define $\epsilon_i : A \to F$ by $\epsilon_i(a) = a^i$. Then $V_{\epsilon_i} = \mathrm{span}\{e_i\}$ and $V = V_{\epsilon_1} \oplus V_{\epsilon_2} \ldots \oplus V_{\epsilon_n}$.

Proof. By definition,

$$\begin{aligned} V_{\epsilon_i} &= \{v \in V : a(v) = \lambda(a)v \text{ for all } a \in A\} \\ &= \{v \in V : a(v) = a^i v \text{ for all } a \in A\} \end{aligned}$$

Now we compute $a(v)$:

$$a(v) = \begin{pmatrix} a^1 & 0 & \ldots \\ 0 & a^2 & \ldots \\ \vdots & \vdots & \\ 0 & \ldots & a^n \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} = \begin{pmatrix} a^1 v^1 \\ a^2 v^2 \\ \vdots \\ a^n v^n \end{pmatrix}$$

For $v \in V_{\epsilon_i}$, $a(v) = a^i v$. So we have

$$\begin{pmatrix} a^1 v^1 \\ a^2 v^2 \\ \vdots \\ a^n v^n \end{pmatrix} = \begin{pmatrix} a^i v^1 \\ a^i v^2 \\ \vdots \\ a^i v^n \end{pmatrix}$$

The only v for which this holds for all $a^1, a^2, \dots a^n$ is

$$v = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v^i \\ \vdots \\ 0 \end{pmatrix}$$

Thus $V_{\epsilon_i} = \operatorname{span}\{e_i\}$. Since $V = \operatorname{span}\{e_1, e_2 \dots e_n\}$,

$$\begin{aligned} V &= \operatorname{span}\{e_1\} \oplus \operatorname{span}\{e_2\} \dots \oplus \operatorname{span}\{e_n\} \\ &= V_{\epsilon_1} \oplus V_{\epsilon_2} \dots \oplus V_{\epsilon_n} \end{aligned}$$

□

Proposition 10.3 (Exercise 5.2)**.** *Let $V = F^n$ and let $A = \mathrm{b}(n, F)$. Then $e_1 = (1, 0, \dots 0) \in V$ is an eigenvector of A with eigenvalue 0. Additionally, then linear map $\lambda : A \to F$ defined by $\lambda(a) = 0$ is a weight for A and the corresponding weight space is $V_\lambda = \operatorname{span}\{e_1\}$.*

Proof. Let $a \in A$. Then

$$a(e_1) = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots \\ 0 & 0 & a_{23} & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix} = 0e_1$$

thus e_1 is an eigenvector of A with eigenvalue 0. Let $\lambda : A \to F$ be defined by $\lambda(a) = 0$. Then

$$V_\lambda = \{v \in V : a(v) = \lambda(a)v \text{ for all } a \in A\}$$

is non-empty since $e_1 \subseteq V_\lambda$. Thus λ is a weight for A. Specifically,

$$V_\lambda = \{v \in V : a(v) = 0 \text{ for all } a \in A\}$$

so for $v \in V_\lambda$, we have

$$a(v) = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots \\ 0 & 0 & a_{23} & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} = \begin{pmatrix} v^2 a_{12} + v^3 a_{13} + \dots + v^n a_{1n} \\ v^3 a_{23} + v^4 a_{24} + \dots + v^n a_{2n} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

thus $v^2, v^3 \dots v^n = 0$, so $V_\lambda = \operatorname{span}\{e_1\}$. □

Definition 10.4. *Let V be a vector space and let $a \in \mathrm{gl}(V)$. The* ***centraliser of*** *a in $\mathrm{gl}(V)$ is*

$$CL_a = \{x \in \mathrm{gl}(V) : a \circ x = x \circ a\}$$

We show in the next lemma that CL_a is a subalgebra of $\mathrm{gl}(V)$.

Lemma 10.5 (Lemma for Exercise 5.3)**.** *CL_a is a subalgebra of* $\mathrm{gl}(V)$*.*

Proof. We need to show that for $x, y \in CL_a$, $[x, y] \in CL_a$.

$$a \circ [x, y] = a(xy - yx) = axy - ayx = xay - yax = xya - yxa = [x, y] \circ a$$

Thus $[x, y] \in CL_a$. □

Proposition 10.6 (Exercise 5.3)**.** *The result in 5.3 (page 39) is a specific case of Lemma 5.4.*

Proof. Let $a, b \in \mathrm{gl}(V)$, so $a, b : V \to V$ are linear and suppose that $a \circ b = b \circ a$. Let $L = CL_a$. As shown, L is a subalgebra of $\mathrm{gl}(V)$, and by definition, $b \in L$.

Let $A = \mathrm{span}\{a\}$. We claim that A is an ideal of L. Clearly $a \in L$, since $a \circ a = a \circ a$. For $\lambda a \in A$ and $x \in L$,

$$\begin{aligned} a \circ [\lambda a, x] &= a(\lambda ax - \lambda xa) = \lambda(a^2 x - axa) = \lambda(axa - axa) = 0 \\ [\lambda a, x] \circ a &= \lambda(axa - xaa) = \lambda(axa - axa) = 0 \end{aligned}$$

Thus A is an ideal of L since $[\lambda a, x] \in A$. Now let

$$W = \{v \in V : x(v) = 0 \text{ for all } x \in A\}$$

We claim that $W = \ker a$. Let $v \in W$. Then $x(v) = 0$ for all $x \in A$, and since $a \in A$, $a(v) = 0$. Thus $v \in \ker a$, so $W \subseteq \ker a$. Now let $v \in \ker a$. Then $a(v) = 0$, so $\lambda a(v) = 0$, so $(\lambda a)(v) = 0$ so $x(v) = 0$ for all $a \in A$. Thus $\ker a \subseteq W$.

Thus by Lemma 5.4, $W = \ker a$ is an L-invariant subspace of V, and since $b \in L$, $b(W) = b(\ker a) = \ker a$. This is precisely the result in 5.3, so that result is a special case of Lemma 5.4 □

Proposition 10.7 (Exercise on page 40)**.** *Let V be a vector space over F and let $a, b : V \to V$ be linear maps such that $a \circ b = b \circ a$. Let $\lambda \in F$, and let $V_\lambda = \{v \in V : a(v) = \lambda v\}$. Then $b(V_\lambda) \subseteq V_\lambda$.*

Proof. Suppose that $x \in b(V_\lambda)$. Then $x = b(\lambda v)$ for some $v \in V_\lambda$. Then so $b \circ a(v) = \lambda b(v)$ so $a(bv) = \lambda(bv)$ so $x \in V_\lambda$, so $b(V_\lambda) \subseteq V_\lambda$. □

Proposition 10.8 (Exercise 5.4i)**.** *Let L be a subalgebra of* $\mathrm{gl}(V)$*. Suppose there is a basis β for V such that every $x \in L$ is represented by a strictly upper triangular matrix with respect to β. Then L is isomorphic to a subalgebra of* $\mathrm{n}(n, F)$ *and hence L is nilpotent.*

Proof. We have the usual map $[\,] : L \to \mathrm{n}(n, F)$ where $[x]$ is the matrix of x with respect to β. $[\,]$ maps into $\mathrm{n}(n, F)$ by hypothesis. We know that M is linear, that is,

$$[ax + y] = a[x] + [y]$$

$[\,]$ also preserves brackts, because it is linear:

$$[[x, y]] = [xy - yx] = [x][y] - [y][x] = [[x], [y]]$$

Thus [ ] is a homomorphism. The kernel of M is the zero map, so by the 1st Isomorphism Theorem,

$$L/\ker[\ ] \cong [L] \implies L \cong [L]$$

and since [ ] is a homomorphism, $[L]$ is a subalgebra of $\mathrm{n}(n, F)$. Thus L is isomorphic to a subalgebra of $\mathrm{n}(n, F)$. Since $\mathrm{n}(n, F)$ is nilpotent, any subalgebra is also nilpotent, so L is nilpotent. □

Proposition 10.9 (Exercise 5.4ii). *Let L be a subalgebra of* $\mathrm{gl}(V)$*. Suppose there is a basis β of V such that all $x \in L$ are represented by upper triangular matrices with respect to β. Then L is isomorphic to a subalgebra of* $\mathrm{b}(n, F)$ *and hence L is solvable.*

Proof. Again we use the homomorphism $[\ ] : L \to \mathrm{b}(n, F)$. [ ] maps into $\mathrm{b}(n, F)$ by hypothesis. By 5.4i, [ ] is a homorphism, so $L \cong [L]$ and $[L] \subseteq \underset{-}{(}n, F)$ is a subalgebra. Thus L is isomorphic to a subalgebra of $\mathrm{b}(n, F)$, so L is solvable. □

Proposition 10.10 (Exercise 5.6i). *Let L be a Lie algebra and let $A \subseteq L$ be a subalgebra. Define*

$$N_L(a) = \{x \in L : [x, a] \in A \text{ for all } a \in A\}$$

Then $N_L(A)$ is a subalgebra of L and $A \subseteq N_L(A)$.

Proof. First we show that $A \subseteq N_L(A)$. Let $a \in A$. Since A is a subalgebra, $[a, b] \in A$ for all $b \in A$. Thus by definition of $N_L(A)$, $a \in N_L(A)$.

Now we show that $N_L(A)$ is a subalgebra of L. Suppose $y, z \in N_L(A)$ and let $a \in A$. We need to show that $[[y, z], a] \in A$. Using the Jacobi identity,

$$[[y, z], a] = -[a, [y, z]] = [y, [z, a]] + [z, [a, y]]$$

Since $y, z \in N_L(A)$, $[z, a], [a, y] \in A$. Thus $[y, [z, a]], [z, [a, y]] \in A$, so $[[y, z], a] \in A$. Thus $[y, z] \in N_L(A)$, so $N_L(A)$ is a subalgebra of L. □

Proposition 10.11 (Exercise 5.6i). *Let L be a Lie algebra and $A \subseteq L$ be a subalgebra. Let $B \subseteq L$ be a subalgebra such that $A \subseteq B \subseteq L$ and A is an ideal of B. Then $B \subseteq N_L(A)$. (Thus $N_L(A)$ is the largest subalgebra of L in which A is an ideal.)*

Proof. We need to show that for $b \in B$, we have $b \in N_L(A)$. Let $b \in B, a \in A$. Since A is an ideal of B, $[a, b] \in A$, so $[b, a] \in A$. Then by definition of $N_L(A), b \in N_L(A)$. □

Proposition 10.12 (Exercise 5.1ii). *Let $L = \mathrm{gl}(n, \mathbb{C})$ and let A be the subalgebra of diagonal matrices. Then $N_L(A) = A$.*

Proof. We know that $A \subseteq N_L(A)$, so we just need to show that $N_L(A) \subseteq A$. Let $x = (x_{ij}) \subseteq N_L(A)$. By the Invariance Lemma, any weight space of A is $N_L(A)$-invariant, that is, for any weight space V_λ of A, $x(V_\lambda) \subseteq V_\lambda$. As shown in Exercies 5.1ii, span$\{e_i\}$ is a weight space for $i = 1, 2, \dots n$. (Recall that $\{e_i\}$ is the standard basis for $\mathbb{C}^n$.) Thus

$$(x_{ij})e_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} \in \operatorname{span}\{e_1\} \implies x_{k1} = 0 \text{ for } k = 2, 3, \dots n$$

$$(x_{ij})e_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} \in \operatorname{span}\{e_2\} \implies x_{k2} = 0 \text{ for } k = 1, 3, 4, \dots n$$

and we can do this for each e_i. As this demonstrates, $x_{ij} = 0$ for $i \neq j$. Thus x is a diagonal matrix, so $x \in A$. Thus $N_L(A) \subseteq A$, so $N_L(A) = A$. □

Proposition 10.13 (Exercise 5.7)**.** *Let V be a vector space, and let $a, y \in \mathrm{gl}(V)$. Then for any $m \geq 1$ (where $m \in \mathbb{N}$),*

$$ay^m = y^m a + \sum_{k=1}^{m} \binom{m}{k} y^{m-k} a_k$$

where $a_1 = [a, y]$ and $a_k = [a_{k-1}, y]$ for $k \geq 2$.

Proof. This is true for $m = 1$ because

$$ay = ya + ay - ya = ya + [a, y] = ya + \sum_{k=1}^{1} \binom{1}{k} y^{1-k} a_k$$

Now suppose that

$$ay^m = y^m a + \sum_{k=1}^{m} \binom{m}{k} y^{m-k} a_k$$

for some $m \geq 1$. Then

$$\begin{aligned}
ay^{m+1} &= ay^m y \\
&= \left(y^m a + \sum_{k=1}^{m} \binom{m}{k} y^{m-k} a_k \right) y \\
&= y^m ay + \sum_{k=1}^{m} \binom{m}{k} y^{m-k} a_k y \\
&= y^m (ya + [a, y]) + \sum_{k=1}^{m} \binom{m}{k} y^{m-k} (ya_k + [a_k, y]) \\
&= y^{m+1} a + y^m a_1 + \sum_{k=1}^{m} \left(\binom{m}{k} y^{m+1-k} a_k + \binom{m}{k} y^{m-k} a_{k+1} \right) \\
&= y^{m+1} a + y^m a_1 + \sum_{k=1}^{m} \binom{m}{k} y^{m+1-k} a_k + \sum_{k=1}^{m} \binom{m}{k} y^{m-k} a_{k+1}
\end{aligned}$$

Consider the far right summation. We can rewrite this by replacing k with $k-1$.

$$\sum_{k=1}^{m} \binom{m}{k} y^{m-k} a_{k+1} = \sum_{k=2}^{m+1} \binom{m}{k-1} y^{m+1-k} a_k$$

Then we can combine this with the $y^m a_1$ term, since $\binom{m}{0} = 1$.

$$\begin{aligned} y^m a_1 + \sum_{k=2}^{m+1} \binom{m}{k-1} y^{m+1-k} a_k &= \binom{m}{0} y^m a_1 + \sum_{k=2}^{m+1} \binom{m}{k-1} y^{m+1-k} a_k \\ &= \sum_{k=1}^{m+1} \binom{m}{k-1} y^{m+1-k} a_k \end{aligned}$$

Consider the other summation term in our expression for ay^{m+1}. We can tack on a $k = m+1$ term since that term would be zero because $\binom{m}{m+1} = 0$.

$$\begin{aligned} \sum_{k=1}^{m} \binom{m}{k} y^{m+1-k} a_k &= \sum_{k=1}^{m} \binom{m}{k} y^{m+1-k} a_k + \binom{m}{m+1} y^{m+1-(m+1)} a_{m+1} \\ &= \sum_{k=1}^{m+1} \binom{m}{k} y^{m+1-k} a_k \end{aligned}$$

Putting this all together, we get

$$\begin{aligned} ay^{m+1} &= y^{m+1} a + \sum_{k=1}^{m+1} \binom{m}{k} y^{m+1-k} a_k + \sum_{k=1}^{m+1} \binom{m}{k-1} y^{m+1-k} a_k \\ &= y^{m+1} a + \sum_{k=1}^{m+1} \left(\binom{m}{k} + \binom{m}{k-1} \right) y^{m+1-k} a_k \end{aligned}$$

By a standard identity for binomial coefficients (Pascal's Rule),

$$\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$$

Thus

$$ay^{m+1} = \sum_{k=1}^{m+1} \binom{m+1}{k} y^{m+1-k} a_k$$

This completes the induction. $\square$

Proposition 10.14 (Exercise 5.7)**.** *Let V be a vector space, and let $y \in \mathrm{gl}(V)$. Then*

$$(\operatorname{ad} y)^m = \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} y^{m-k} (\operatorname{ad} y)^k$$

Proof. Let $a \in \mathrm{gl}(V)$ and define $a_1 = [a, y]$ and $a_k = [a_{k-1}, y]$ for $k \geq 2$. Then $a_1 = -\operatorname{ad} y(a)$ and $a_k = (-1)^k (\operatorname{ad} y)^k (a)$. By the previous proposition,

$$\begin{aligned}
ay^m &= y^m a + \sum_{k=1}^{m} \binom{m}{k} y^{m-k} a_k \\
y^m a - ay^m &= -\sum_{k=1}^{m} \binom{m}{k} y^{m-k} (-1)^k (\operatorname{ad} y)^k (a) \\
[y^m, a] &= \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} y^{m-k} (\operatorname{ad} y)^k (a)
\end{aligned}$$

Thus the map $(\operatorname{ad} y)^m$ is

$$(\operatorname{ad} y)^m = \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} y^{m-k} (\operatorname{ad} y)^k$$

□

11 Chapter 6 Exercises

Proposition 11.1 (Exercise 6.1i)**.** *Let V be an n-dimensional vector space where $n \geq 1$ and let $x : V \to V$ be a nilpotent linear map. Then there exists a nonzero $v \in V$ such that $x(v) = 0$.*

Proof. Suppose that there is no nonzero $v \in V$ such that $x(v) = 0$. Then $\ker x = \{0\}$, so x is one-to-one. Let β be some basis for V. Since x is one-to-one, $x(\beta)$ is linearly independent, so $x(\beta)$ is a basis for V. Thus $x(V) = V$. Then by induction $x^r(V) = V$ for all $r \in \mathbb{N}$, so x is not nilpotent. This contradicts the hypothesis, so we conclude that there must be some nonzero $v \in V$ such that $x(v) = 0$. $\square$

Lemma 11.2 (for Exercise 6.1ii)**.** *Let V be a finite-dimensional vector space and let $x : V \to V$ be a nilpotent linear map. Then $\operatorname{rank} x < \dim V$.*

Proof. By Exercise 6.1i, there exists a non-zero $v \in V$ such that $x(v) = 0$. Thus $\dim \ker x > 0$. Thus by the Rank-Nullity Theorem, $\dim \operatorname{im} x = \operatorname{rank} x < \dim V$. $\square$

Lemma 11.3 (for Exercise 6.1ii)**.** *Let V be a 1-dimensional vector space and let $x : V \to V$ be a nilpotent linear map. Thus $x(v) = 0$ for all $v \in V$.*

Proof. By the previous lemma, $\operatorname{rank} x < 1$ so $\operatorname{rank} x = 0$. Thus $\dim \operatorname{im} x = 0$, so $x(v) = 0$ for all $v \in V$. $\square$

Lemma 11.4 (for Exercise 6.1ii)**.** *Let V be an n-dimensional vector space over F where $n \geq 1$. Let $x : V \to V$ be a nilpotent linear map. Let U be a subspace of V. Define $\bar{x} : V/U \to V/U$ by*

$$\bar{x}(w + U) = x(w) + U$$

for $w + U \in V/U$. Then $\bar{x}$ is a nilpotent linear map. (We refer to $\bar{x}$ as the map induced by x.)

Proof. One can check that $\bar{x}$ is linear using the definitions of addition and scalar multiplication in V/U. We show that $\bar{x}$ is nilpotent. We know that x is nilpotent, so there exists $r \in \mathbb{N}$ such that $x^r(v) = 0$ for all $v \in V$. Then

$$\bar{x}^r(w + U) = x^r(w) + U = 0 + U = U$$

Thus $\bar{x}$ is nilpotent. $\square$

Proposition 11.5 (Exercise 6.1ii)**.** *Let V be an n-dimensional vector space and let $x : V \to V$ be a nilpotent linear map. Let $v \in V$ such that $v \neq 0$ and $x(v) = 0$. Let $U = \operatorname{span}\{v\}$. Define $\bar{x} : V/U \to V/U$ as above. Then there is a basis $\{v_1 + U, \dots v_{n-1} + U\}$ of V/U such that the matrix of $\bar{x}$ in this basis is strictly upper triangular.*

Proof. We begin with the case $n = 2$, so we assume V is 2-dimensional. Let v_1 be any vector in $V - U$. Then $v_1 \neq 0$ and $v_1 \notin U$ thus $v_1 + U \neq U$. Thus $v_1 + U$ is a non-zero element of V/U, and since V/U is one-dimensional, $V/U = \operatorname{span}\{v_1 + U\}$. Since $\bar{x}$ is a nilpotent linear map from a 1-dimensional vector space to itself, $\bar{x}$ is the zero map. Thus in any basis its matrix representation is $[x] = (0)$ which is strictly upper triangular.

Now we proceed by induction on n. We suppose that for any k-dimensional vector space V, and any nilpotent linear map $x : V \to V$, there is a basis $\{v_1 + U, \dots v_{k-1} + U\}$ of V/U in which the matrix of $\bar{x}$ is upper triangular (where $U = \text{span}\{v\}$ for some $v \neq 0$ with $x(v) = 0$).

Let W be a $(k+1)$-dimensional vector space and let $y : W \to W$ be a nilpotent linear map. By Exercise 6.1i, there exists a nonzero $w \in W$ such that $y(w) = 0$. We let $B = \text{span}\{w\}$. Then W/B is a k-dimsional vector space, so by inductive hypothesis there is a basis □

Proposition 11.6 (Exercise 6.1ii, Base Case for Induction)**.** *Let V be an 2-dimensional vector space and let $x : V \to V$ be a nilpotent linear map. Let $v \in V$ such that $v \neq 0$ and $x(v) = 0$. Let $U = \text{span}\{v\}$. Define $\bar{x} : V/U \to V/U$ as above. Then there is a basis $\{v_1 + U\}$ of V/U such that the matrix of $\bar{x}$ in this basis is strictly upper triangular.*

Proof. Let v_1 be any vector in $V - U$. Then $v_1 \neq 0$ and $v_1 \notin U$ thus $v_1 + U \neq U$. Thus $v_1 + U$ is a non-zero element of V/U, and since V/U is one-dimensional, $V/U = \text{span}\{v_1 + U\}$. Since $\bar{x}$ is a nilpotent linear map from a 1-dimensional vector space to itself, $\bar{x}$ is the zero map. Thus in any basis its matrix representation is $[x] = (0)$ which is strictly upper triangular. □

Proposition 11.7 (Exercise 6.1ii, Inductive Step for Induction)**.** *Suppose that for every k-dimensional vector space V with a nilpotent linear map $x : V \to V$ and $U = \text{span}\{v\}$ for some $v \in V$ with $x(v) = 0$ and $v \neq 0$, there is a basis $\{v_1 + U, \dots, v_{n-1} + U\}$ of V/U in which $[\bar{x}]$ is strictly upper triangular. Then let W be a $k+1$ dimensional vector space with $y : W \to W$ a nilpotent map with $y(w) = 0$ for some $w \in W$ with $w \neq 0$ and $B = \text{span}\{w\}$. Then there is a basis of W/B in which $[\bar{y}]$ is strictly upper triangular.*

Proof. Let W be such a space. Then W/B is k-dimensional, so there is a basis of W/B in which $[\bar{\bar{y}}]$ is strictly upper triangular. Then by the previous proposition, this basis with w added gives a basis of W in which $[\bar{y}]$ is strictly upper triangular. □

Proposition 11.8 (Exercise 6.2ii)**.** *Let V be an n-dimensional complex vector space and let $x : V \to V$ be a linear map. Let v be an eigenvector of x with corresponding eignevalue λ. Let $U = \text{span}\{v\}$. Define $\bar{x} : V/U \to V/U$ by*

$$\bar{x}(w + U) = x(w) + U$$

The map $\bar{x}$ is linear.

Proof. Let $a \in \mathbb{C}$, $w_1, w_2 \in V$.

$$\begin{aligned}
\bar{x}(a(w_1 + U) + (w_2 + U)) &= \bar{x}((aw_1 + w_2) + U) \\
&= x(aw_1 + w_2) + U \\
&= (ax(w_1) + x(w_2)) + U \\
&= a(x(w_1) + U) + (x(w_2) + U) \\
&= a\bar{x}(w_1 + U) + \bar{x}(w_2 + U)
\end{aligned}$$

□

Proposition 11.9 (Exercise 6.2ii)**.** *Let V be an n-dimensional complex vector space and let $x : V \to V$ be a linear map. Let v be an eigenvector of x and let $U = \operatorname{span} v$. Let $\bar{x} : V/U \to V/U$ be the induced map and let $\beta = \{v_1 + U, \ldots v_{n-1} + U\}$ be a basis of V/U such that $[\bar{x}]_\beta$ is upper triangular. Then $\gamma = \{v, v_1, \ldots v_{n-1}\}$ is a basis of V such that $[x]_\gamma$ is upper triangular.*

Proof. First we show that γ is linearly independent. We have the canonical map $\pi : V \to V/U$ by $\pi(w) = w + U$. Suppose there are scalars $a, a^1, \ldots a^{n-1} \in \mathbb{C}$ such that

$$av + \sum_{i=1}^{n-1} a^i v_i = 0$$

Then (using Einstein summation notation)

$$0 = \pi(av + a^i v_i) = a\pi(v) + a^i \pi(v_i)$$

But $\pi(v) = 0$ so we have

$$a^i \pi(v_i) = 0$$

By hypothesis, $\{\pi(v_i)\}_{i=1}^{n-1}$ is a basis for V/U so it is linearly independent. Thus $a^i = 0$ for $i = 1, 2, \ldots n-1$. Returning to the original equation, we now have $av = 0$. Since v is an eigenvector, it is not the zero vector, so $a = 0$. Thus γ is linearly independent, so it forms a basis for V. □

Lemma 11.10. *Let V be an n-dimensional vector space and let $x : V \to V$ be a nilpotent linear map. Then $x^n(v) = 0$ for all $v \in V$.*

Proof. By the lemma for Exercise 6.1ii, the rank of x is strictly less than the dimension of V (unless V is zero-dimensional), so

$$\dim V > \dim x(V) > \dim x^2(V) > \ldots > \dim x^n(V)$$

where there are $n+1$ such inequalities. But a string of $n+1$ inequalities involving integers means that $\dim x^n(v) = 0$. □

Proposition 11.11 (Exericse 6.3)**.** *Let L be a nilpotent complex Lie algebra. Then every 2-dimensional subalgebra of L is abelian.*

Proof. By Engel's Theorem (Theorem 6.3), for every $x \in L$, the map $\operatorname{ad} x : L \to L$ is nilpotent. Let V be a 2-dimensional subalgebra of L. Either V is abelian, or we can choose a basis $\{x, y\}$ of V such that $[x, y] = x$. In the latter case, we know that $(\operatorname{ad} y)^2 = 0$, so

$$0 = [y, [y, x]] = -[y, [x, y]] = -[y, x] = [x, y]$$

Thus even if V is "not abelian," all the brackets in V are zero, so V is abelian. □

Lemma 11.12. *For $i, j, k, p \in \mathbb{N}$,*

$$\sum_{k=1}^{p} \delta_{ik}\delta_{kj} = \delta_{ij}$$

Proof. See Wikipedia. □

(Exercise 6.4)
Erdmann and Wildon define matrices $x, y \in \mathrm{gl}(p, \mathbb{Z}_p)$. We equivalently view x and y in terms of their ijth position as

$$\begin{aligned}(x)_{ij} &= \delta_{j,i+1} + \delta_{ip}\delta_{1j} \\ (y)_{ij} &= \delta_{ij}(i-1) = \delta_{ij}(j-1)\end{aligned}$$

Then the matrix products xy and yx are given by

$$\begin{aligned}(xy)_{ij} &= \sum_{k=1}^{p} x_{ik}y_{kj} \\ &= \sum_{k=1}^{p}(\delta_{k,i+1} + \delta_{ip}\delta_{1k})\delta_{kj}(j-1) \\ &= (j-1)\left(\sum_{k=1}^{p}\delta_{k,i+1}\delta_{k,j} + \sum_{k=1}^{p}\delta_{ip}\delta_{1k}\delta_{kj}\right) \\ &= (j-1)(\delta_{i+1,j} + \delta_{1j}\delta_{ip}) \\ (yx)_{ij} &= \sum_{k=1}^{p} y_{ik}x_{kj} \\ &= \sum_{k=1}^{p}\delta_{ik}(i-1)(\delta_{j,k+1} + \delta_{kp}\delta_{1j} \\ &= (i-1)\left(\sum_{k=1}^{p}\delta_{ik}\delta_{j,k+1} + \sum_{k=1}^{p}\delta_{ik}\delta_{kp}\delta_{1j}\right) \\ &= (i-1)(\delta_{i+1,j} + \delta_{ip}\delta_{1j})\end{aligned}$$

Then we can compute $[x, y]$ as

$$\begin{aligned}[x,y]_{ij} &= (xy)_{ij} - (yx)_{ij} = (j-1)(\delta_{i+1,j} + \delta_{1j}\delta_{ip}) - (i-1)(\delta_{i+1,j} + \delta_{ip}\delta_{1j}) \\ &= (j-i)(\delta_{i+1,j} + \delta_{ip}\delta_{1j}) \\ &= (j-i)\delta_{i+1,j} + (j-i)\delta_{ip}\delta_{1j} \\ &= \delta_{i+1,j} + (1-p)\delta_{ip}\delta_{1j} \\ &= \delta_{i+1,j} + \delta_{ip}\delta_{1j} \\ &= x_{ij}\end{aligned}$$

Note that in the last few equalities, the factor $j - i$ becomes 1 since $\delta_{i+1,j}$ is zero unless $i + 1 = j$, and $(1 - p) = 1$ since the field is $\mathbb{Z}_p$. Thus $[x, y] = x$.

Proposition 11.13 (Exericse 6.4)**.** *As defined above x, y span a 2-dimensional solvable subalgebra of* $\mathrm{gl}(p, F)$*.*

Proof. Since $[x, y] = x$, we know that $\text{span}\{x, y\}$ is closed under brackets, so it is a subalgebra of $\text{gl}(p, F)$. We know that all 2-dimensional Lie algebras are solvable since the non-abelian 2-dimensional Lie algebra has one-dimensional (and hence abelian) derived algebra. □

Proposition 11.14 (Exercise 6.4)**.** *As defined above, the matrices x, y have no common eigenvector.*

Proof. Let $v = (v_1, v_2, \dots v_p)$. Then

$$xv = (v_2, v_3 \dots v_p, v_1)$$

So if v is an eigenvector of x, then

$$\begin{aligned} v_2 &= \lambda v_1 \\ v_3 &= \lambda v_2 \\ &\vdots \\ v_1 &= \lambda v_p \end{aligned}$$

so $\lambda^p v_1 = v_1$. Thus the eigenvalues of x are the pth roots of unity,

$$\begin{aligned} \lambda &\in \{e^{2\pi ik/p} : k = 0, 1, \dots (p-1)\} \\ &\in \{1, e^{2\pi i/p}, e^{4\pi i/p}, \dots\} \end{aligned}$$

The eigenvector corresponding to each λ is

$$v_\lambda = (1, \lambda, \lambda^2, \lambda^3, \dots \lambda^{p-1})$$

Now we compute the eigenvectors for y.

$$yv = (0, v_2, 2v_3, \dots (p-1)v_p)$$

Eigenvalues of y are the diagonal entries, $\lambda \in \{0, 1, 2, \dots (p-1)\}$, and the corresponding eigenvectors are the standard basis vectors for F^n. (Recall that F is a field of characteristic p.)

$$v_\lambda = e_\lambda = (0, 0, \dots, 1, \dots 0, 0)$$

If v is an eigenvector of x, then it has all nonzero entries, but then it could not be an eigenvector of y. Thus x, y have no common eigenvectors. □

Because x and y have no common eigenvector, this example demonstrates that the hypothesis that the field be complex in Proposition 6.6 is necessary. If $V = F^n$ where F is a field of characteristic p, then we have shown that $L = \text{span}\{x, y\}$ is a solvable subalgebra of $\text{gl}(p, F) \cong \text{gl}(V)$, and x, y have no common eigenvector.

Proposition 11.15 (Exercise 6.4)**.** *Let x be the matrix defined above. Then $x^p = I_p$, where I_p is the identity matrix.*

Proof. We compute the characteristic equation of x.

$$x - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & -\lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & -\lambda & \dots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & -\lambda & 1 \\ 1 & 0 & 0 & \dots & 0 & -\lambda \end{pmatrix}$$

We compute the determinant of $x - \lambda I$ by expansion by cofactors along the top row. Thankfully, all but two of the entries are zero.

$$\det(x - \lambda I) = -\lambda \det \begin{pmatrix} -\lambda & 1 & 0 & \dots & 0 \\ 0 & -\lambda & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & -\lambda & 1 \\ 0 & 0 & \dots & 0 & -\lambda \end{pmatrix} - \det \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & -\lambda & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & -\lambda & 1 \\ 1 & 0 & \dots & 0 & -\lambda \end{pmatrix}$$

The first matrix here is diagonal, so its determinant can be read off as the product along the diagonal, $(-\lambda)^{p-1}$. To compute the determinant of the second matrix, we will iteratively expand along the second column. Notice that when we expand it along the second column, we get the same matrix, except one dimension smaller, and we multiply by (-1).

$$\det \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & -\lambda & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & -\lambda & 1 \\ 1 & 0 & \dots & 0 & -\lambda \end{pmatrix}_{p-1} = -\det \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & -\lambda & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & -\lambda & 1 \\ 1 & 0 & \dots & 0 & -\lambda \end{pmatrix}_{p-2}$$

After doing this expansion $p - 3$ times, we get

$$\det \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & -\lambda & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & -\lambda & 1 \\ 1 & 0 & \dots & 0 & -\lambda \end{pmatrix}_{p-1} = (-1)^{p-3} \det \begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix} = (-1)^{p-2}$$

Thus the characteristic polynomial of x is

$$\begin{aligned} &\det(x - \lambda I) = (-\lambda)^p - (-1)^{p-2} = (-1)^p \lambda^p - (-1)^p = (-1)^p(\lambda^p - 1) = 0 \\ &\implies \lambda^p - 1 = 0 \end{aligned}$$

By the Cayley-Hamilton Theorem, every square matrix over a commutative ring satisfies its own characteristic polynomial. Thus

$$x^p - I_p = 0 \implies x^p = I_p$$

□

The matrices x, y also illustrate that the requirement that the field by complex in the following proposition (Exercise 6.5i) is necessary. Let F be a field of characteristic p, let $V = F^n$, and let x, y be the matrices above. Let $L = \operatorname{span}\{x, y\}$. Then as shown above, L is a solvable subalgebra of $\mathrm{gl}(V)$, so all the other hypotheses of 6.5i are satisfied. However, $L' = \operatorname{span}\{x\}$, and x is not nilpotent, since $x^p = I_p$. Thus the theorem fails without the hypothesis that the underlying field by $\mathbb{C}$.

Proposition 11.16 (Exericse 6.5i)**.** *Let V be a complex vector space and let $L \subseteq \mathrm{gl}(V)$ be a solvable subalgebra. Then every element of L' is nilpotent.*

Proof. By Lie's Theorem (Theorem 6.5), there is a basis β of V in which every element of L is represented by an upper triangular matrix. For $z \in L'$, we can write z as a linear combination of brackets,

$$z = a^i[x_i, y_i]$$

where $a^i \in \mathbb{C}$ and $x_i, y_i \in L$. x_i, y_i have upper triangular matrices in the basis β, $[x_i, y_i]$ has a strictly upper triangular matrix (in β). Thus z has a strictly upper triangular matrix representation, so it is a nilpotent map. □

Proposition 11.17 (Exercise 6.5ii)**.** *Let L be a complex Lie algebra. Then L is solvable if and only if L' is nilpotent.*

Proof. First suppose that L' is nilpotent. Then L' is solvable, and so $L'^m = 0$ for some m, and thus $L^{(m+1)} = 0$, so L is solvable.

Now we suppose that L is solvale and show that L' is nilpotent. Using the adjoint homomorphism $\mathrm{ad} : L \to \mathrm{gl}(L)$, we can see that $\mathrm{ad}\, L$ is a subalgebra of $\mathrm{gl}(L)$, and by Lemma 4.4, $\mathrm{ad}\, L$ is solvable since it is a homomorphic image of L. Thus by Lie's Theorem (Theorem 6.5), there is a basis β of L such that every element of $\mathrm{ad}\, L$ is represented by an upper triangular matrix.

We claim that for $z \in L'$, $\mathrm{ad}\, z$ is nilpotent. If $z \in L'$, we can write z as a linear combination of brackets,

$$z = a^i[x_i, y_i]$$

where $a^i \in \mathbb{C}$ and $x_i, y_i \in L$. Then since ad is a homomorphism,

$$\mathrm{ad}\, z = \mathrm{ad}(a^i[x_i, y_i]) = a^i[\mathrm{ad}\, x_i, \mathrm{ad}\, y_i]$$

Since $\mathrm{ad}\, x_i, \mathrm{ad}\, y_i$ have upper triangular matrices in the basis β, $[\mathrm{ad}\, x_i, \mathrm{ad}\, y_i]$ has a strictly upper triangular matrix (in β). Thus $\mathrm{ad}\, z$ is nilpotent. Thus by Engel's Theorem (2nd version), L' is nilpotent. □

Proposition 11.18 (Exercise 6.6)**.** *Let V be an n-dimensional complex vector space and let $x, y : V \to V$ be linear maps such that*

$$x \circ [x, y] = [x, y] \circ x$$
$$y \circ [x, y] = [x, y] \circ y$$

Then $[x, y]$ is a nilpotent map.

Proof. Let $L = \text{span}\{x, y, [x, y]\}$. We claim that L is a solvable subalgebra of $\text{gl}(V)$. To see that L is a subalgebra, note that

$$\begin{aligned}
[x, y] &\in L \\
[x, [x, y]] &= x \circ [x, y] - [x, y] \circ x = 0 \in L \\
[y, [x, y]] &= y \circ [x, y] - [x, y] \circ y = 0 \in L
\end{aligned}$$

To see that L is solvable, note that $L' = \text{span}\{[x, y]\}$ which is abelian since it is one-dimensional, so $L'' = 0$.

Since L is a solvable subalgebra of $\text{gl}(V)$, by Lie's Theorem (Theorem 6.5), there is a basis of V in which every element of L is represented by an upper triangular matrix. Let $M_x, M_y, M_{[x,y]}$ be the matrices of x, y, and $[x, y]$ respectively. Because M is a homomorphism,

$$M_{[x,y]} = M_{xy-yx} = M_x M_y - M_y M_x = [M_x, M_y]$$

As shown in Exercise 4.5i, the commutator of two upper triangular matrices is strictly upper triangular, so since M_x, M_y are upper triangular, $M_{[x,y]}$ is strictly upper triangular. It is a standard result that strictly upper triangular matrices are nilpotent, and that a linear map is nilpotent if and only if its matrix is nilpotent. Thus $[x, y]$ is nilpotent. $\square$

12 Chapter 7 Exercises

(Exercise 7.1)

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[e, f] = h \qquad [f, h] = 2f \qquad [h, e] = 2e$$

$$[\operatorname{ad} h] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad [\operatorname{ad} e] = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad [\operatorname{ad} f] = \begin{pmatrix} 0 & 0 & 2 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Proposition 12.1 (Exercise 7.2)**.** *Let V be an L-module. Define $\phi : L \to \mathrm{gl}(V)$ by $\phi(x)(v) = x \cdot v$. Then ϕ is a Lie algebra homomorphism.*

Proof. Linearity of ϕ follows immediately from the M2 axiom for L-modules. Let $x, y \in L, v \in V$.

$$\begin{aligned} \phi([x, y])(v) &= [x, y] \cdot v \\ &= x \cdot (y \cdot v) - y \cdot (x \cdot v) \\ &= \phi(x) \circ \phi(y)(v) - \phi(y) \circ \phi(x)(v) \\ &= [\phi(x), \phi(y)](v) \end{aligned}$$

□

Proposition 12.2 (Exericise 7.3)**.** *Let L be a Lie algebra and let V be an L-module. Then V is irreducible if and only if for any non-zer $v \in V$ the submodule generated by v contains all elements of V.*

Proof. First, suppose that V is an L-module such that for any non-zero v, the submodule generated by v is V. We will show that any non-zero submodule of V is equal to V. Let $W \subseteq V$ be a non-zero submodule. Then there exists some non-zero $v \in W$. Because W is a submodule, any product of the form

$$x_1 \cdot (x_2 \cdot \ldots \cdot (x_m \cdot v) \ldots)$$

is inside of W. By hypothesis, products of this form span V for any non-zero $v \in V$. Thus $W = V$. Thus V has non non-zero proper submodules, so V is irreducible.

Now suppose that V is irreducible. Define U to be the submodule generated by v, that is,

$$U = \operatorname{span}\{x_1 \cdot (x_2 \cdot \ldots \cdot (x_m \cdot v) \ldots) : x_i \in L\}$$

Then we know that U is a non-zero submodule of V. Since V is irreducible, this means that $U = V$. □

Proposition 12.3 (Exercise 7.5)**.** *Let L be a finite-dimensional Lie algebra. Let* $\mathrm{ad} : L \to \mathrm{gl}(L)$ *be the adjoint homomorphism, and define an action of L on itself by*

$$L \times L \to L \qquad x \cdot y = \mathrm{ad}(x)(y) = [x, y]$$

Then the submodules of L as a module are precisely the ideals of L.

Proof. Let $I \subseteq L$ be any subset, and let $a \in I, x \in L$. Then

$$x \cdot a \in I \iff [x, a] \in I$$

so I is L-invariant exactly when I is an ideal. $\square$

Proposition 12.4 (Exercise 7.6i)**.** *Let F be a field and let $L = \mathrm{b}(n, F)$ and $V = F^n$. Then V is an L-mocule where the action is*

$$L \times V \to V \qquad (x, v) \mapsto xv$$

that is, multiplying the matrix by a column vector.

Proof. Let $a, b \in F, v, w \in V$, and $x, y \in L$. Using standard properties of matrix multiplication,

$$\begin{aligned}
(ax + by)v &= a(xv) + b(yv) \\
x(av + bw) &= xav + xbw = a(xv) + b(xw) \\
[x, y]v &= (xy - yx)v = x(yv) - y(xv)
\end{aligned}$$

$\square$

Proposition 12.5 (Exercise 7.6ii)**.** *Let F be a field, and let $L = \mathrm{b}(n, F), V = F^n$. Let $\{e_1, e_2, \ldots e_n\}$ be the standard basis for F^n, and let $W_r = \mathrm{span}\{e_1, e_2, \ldots e_r\}$. Then W_r is a submodule of V (where V has the same module structure as in part i).*

Proof. To show: for $x \in L, w \in W_r$, we have $xw \in W_r$. Let $x \in L, w \in W_r$. Let x_{ij} be the ijth entry of x and w_i be the ith entry of w. Since x is upper triangular, $x_{ij} = 0$ for $j < i$ and since $w \in W_r$, $w_i = 0$ for $r < i$. We know that

$$(xw)_{ij} = \sum_{j=1}^{n} x_{ij} w_j$$

When $i > r$, there are two possibilities: $j \leq r$ or $j > r$. If $j \leq r$, then $j \leq r < i$ so $x_{ij} = 0$. If $j > r$, then $w_j = 0$. Thus when $i > r$, each term of the summation is zero, so $(xw)_i = 0$ for $i < r$. Thus $xw \in W_r$. $\square$

Proposition 12.6 (Exercise 7.6iii)**.** *Let F be a field, let $V = F^n$, and let $L = \mathrm{b}(n, F)$. Let V be an L-module by applying matrices to column vectors. Then every non-zero submodule of V is equal to some W_r where*

$$W_r = \mathrm{span}\{e_1, e_2, \ldots e_r\}$$

($\mathrm{span}\{e_1, e_2, \ldots e_n\}$ *is the standard basis for F^n.) Furthermore, each W_r is indecomposable, and if $n \geq 2$, then V is not completely reducible as an L-module.*

Proof. We prove the first assertion by induction on r. The base case is $r = 1$. We claim that every 1-dimensional submodule of V is $W_1 = \text{span}\{e_1\}$. Let $W \subseteq V$ be an 1-dimensional submodule, so $W = \text{span}\{w\}$ for some nonzero $w \in V$. Let $w = (w^1, w^2, \ldots w^n)$. Let e_{ij} be the usual matrix basis element. Then since W is a submodule,

$$\begin{aligned} e_{11}w &= (w^1, 0, 0, \ldots) \in W \\ e_{12}w &= (w^2, 0, 0, \ldots) \in W \\ &\vdots \\ e_{1n}w &= (w^n, 0, 0, \ldots) \in W \end{aligned}$$

Since w is nonzero, one of $w^1, w^2, \ldots$ is nonzero, so $\text{span}\{e_1\} \in W$. Since W is a 1-dimensional vector space, it must be equal to $\text{span}\{e_1\}$. Thus any 1-dimensional submodule of V is equal to W_1.

Now for the inductive step. We suppose that every k-dimensional submodule of V is equal to W_k. We will show that this implies that every $(k+1)$-dimensional submodule of V is equal to W_{k+1}. Let U be a $(k+1)$-dimensional submodule of V. Let $U = \text{span}\{u_1, u_2, \ldots u_{k+1}\}$. Then $U' = \text{span}\{u_1, u_2, \ldots u_k\}$ is a k-dimensional submodule, so by the inductive hypothesis, $U' = W_k$. Then $U = W_k \oplus \text{span}\{u_{k+1}\}$. Since U is $(k+1)$-dimensional, it must be that $u_{k+1} \notin W_k$, so u_{k+1} has a non-zero entry after the kth entry.

$$u_{k+1} = (a^1, a^2, \ldots a^k, a^{k+1}, \ldots a^n) \qquad \text{One of } a^{k+1}, a^{k+2}, \ldots a^n \text{ is nonzero.}$$

Then since U is a submodule of V,

$$\begin{aligned} e_{k+1,k+1}u_{k+1} &= (0, \ldots a^{k+1}, 0, \ldots) \in U \\ e_{k+1,k+2}u_{k+1} &= (0, \ldots a^{k+2}, 0, \ldots) \in U \\ &\vdots \\ e_{k+1,n}u_{k+1} &= (0, \ldots a^n, 0, \ldots) \in U \end{aligned}$$

Where each time, the product is in $\text{span}\{e_{k+1}\}$. Since one of $a^{k+1}, a^{k+2}, \ldots a^n$ is nonzero, this implies that $\text{span}\{e_{k+1}\} \in U$. Thus

$$W_{k+1} = W_k \oplus \text{span}\{e_{k+1}\} = \text{span}\{e_1, e_2, \ldots e_k, e_{k+1}\} \subseteq U$$

Since U is a $(k+1)$-dimensional vector space, it follows that $U = W_{k+1}$. This completes the induction.

Now, we show that each W_r is indecomposable and if $n \geq 2$ then V is not completely reducible as an L-module. As shown, the only submodules of V are $W_1, W_2, \ldots W_{n-1}, W_n = V$ and we have proper inclusions

$$W_1 \subset W_2 \subset W_3 \subset \ldots \subset W_{n+1} \subset V$$

Thus no W_r can be written as a direct sum of two submodules of V, since any direct sum is "absorbed," that is, $W_a \oplus W_b = W_{\max(a,b)}$. If $n \geq 2$, then V has at least one nonzero proper submodule, W_1, but no direct sum of submodules is equal to V. $\square$

Proposition 12.7 (Exercise 7.7)**.** *Let L be a Lie algebra, and let V be an n-dimensional L-module with a submodule W of dimension m. Then there exists a basis β of V such that the action of every $x \in L$ is represented by a "block matrix" of the form*

$$[x] = \begin{pmatrix} X_1 & X_2 \\ 0 & X_3 \end{pmatrix}$$

where X_1 is a $m \times m$ matrix. Furthermore, X_1 is the matrix of x restricted to W, and X_3 is the matrix of the action of x on the factor module V/W.

Proof. Let $\alpha = \{w_1, w_2, \ldots w_m\}$ be a basis for W. Extend α to a basis of V, and call this basis β.

$$\beta = \{w_1, w_2, \ldots w_m, v_1, v_2, \ldots v_{n-m}\}$$

Let $x \in L$, and let $[x]_\beta$ be the matrix of x with respect to β. Then for each i with $1 \leq i \leq m$,

$$[x]_\beta [w_i]_\beta = [x \cdot w_i]_\beta$$

Since W is a submodule, $x \cdot w_i \in W$. The multiplication $[x]_\beta [w_i]_\beta$ just picks off the ith column of $[x]_\beta$, so for each column $1, 2, \ldots m$ of $[x]_\beta$, the column lies in W. By construction of β, for any $w \in W$,

$$[w]_\beta = (a^1, a^2, \ldots a^m, 0, 0, \ldots)$$

Thus the bottom left block of $[x]$ is zeroes, as was to be shown.

Now consider the matrix $[x]_\alpha$. As before, the mutliplication

$$[x]_\alpha [w_i]_\alpha$$

picks off the ith column of $[x]_\alpha$, so $[x \cdot w_i]_\alpha$ is the ith column of $[x]_\alpha$. Since nothing in W has a nonzero entry past the mth entry, the block X_2 (of $[x]_\beta$) does not affect anything when x is restricted to W. Thus $[x]_\alpha$ is the upper left block X_1.

Now we show that X_3 is the matrix of x acting on V/W. When x acts on V/W, it acts as the map $v + W \mapsto x \cdot v + W$, so when x acts on $v + W$, it ignores the first m entries of $[v]_\beta$. Thus only the bottom left block X_3 actson V/W. □

Proposition 12.8 (Exercise 7.8)**.** *Let L be the Heisenberg algebra over $\mathbb{C}$, that is, $L = \operatorname{span}\{f, g, z\}$ with $[f, g] = z$ and $[f, z] = [g, z] = 0$. L does not have a faithful finite-dimensional irreducible representation.*

Proof. Let V be a finite-dimensional vector space and let $\phi : L \to \mathrm{gl}(V)$ be an irreducible representation. We will show that ϕ is not faithful. Since $z \in Z(L)$, by Lemma 7.14, $\phi(z) = \lambda I_V$ for some $\lambda \in \mathbb{C}$. (I_V is the identity transformation on V.) Now we compute the trace of $\phi(z)$.

$$\operatorname{tr} \phi(z) = \operatorname{tr}[\phi(f), \phi(g)] = 0$$

Thus $\lambda = 0$. Thus $\phi(z)$ is the zero map, so ϕ has a nonzero kernel. Thus ϕ is not a faithful representation. □

Proposition 12.9 (Exercise 7.9)**.** *Let L be the 2-dimensional complex non-abelian Lie algebra given by $L = \operatorname{span} x, y, [x, y] = x$. Then we define a linear map $\phi : L \to \mathrm{gl}(\mathbb{C}^2)$ by*

$$\phi(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \phi(y) = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

Then ϕ is a representation of L. Furthermore, ϕ is isomorphic to the adjoint representation of L on itself.

Proof. To show that ϕ is a representation, we just need to show that it is a homomorphism, that $\phi([x, y]) = [\phi(x), \phi(y)]$. By routine computations,

$$\phi([x, y]) = \phi(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$[\phi(x), \phi(y)] = \phi(x)\phi(y) - \phi(y)\phi(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Now we find an explicit isomorphism between the adjoint representation, $\mathrm{ad} : L \to \mathrm{gl}(L)$, and ϕ. We define $\theta : \mathbb{C}^2 \to L$ by

$$\theta(e_1) = x \qquad \theta(e_2) = -x + y$$

where $e_1 = (1, 0), e_2 = (0, 1)$ are the standard basis for $\mathbb{C}^2$. We need to show that for $ax + by \in L, v \in \mathbb{C}^2$,

$$\theta(\phi(ax + by)v) = \mathrm{ad}(ax + by)\theta(v)$$

We only need to show this holds for the basis vectors e_1, e_2.

$$\begin{aligned}
\theta(\phi(ax + by)e_1) &= \theta(a\phi(x)e_1 + b\phi(y)e_1) \\
&= \theta(a(0) + b(-e_1)) \\
&= -b\theta(e_1) \\
&= -bx \\
\mathrm{ad}(ax + by)\theta(e_1) &= [ax + by, x] \\
&= a[x, x] + b[y, x] \\
&= -bx \\
\theta(\phi(ax + by)e_2) &= \theta(a\phi(x)e_2 + b\phi(y)e_2) \\
&= \theta(ae_1 + be_1) \\
&= (a + b)x \\
\mathrm{ad}(ax + by)\theta(e_2) &= [ax + by, \theta(e_2)] \\
&= [ax + by, -x + y] \\
&= -a[x, x] + a[x, y] - b[y, x] + b[y, y] \\
&= (a + b)x
\end{aligned}$$

□

Proposition 12.10 (Exercise 7.11)**.** *Let L be a Lie algebra over F, and let $\phi : L \to \mathrm{gl}(1, F)$ be a representation of L. Then $\phi(L') = 0$.*

Proof. If ϕ is the zero map, then clearly $\phi(L') = 0$. If ϕ is not the zero map, then the image of ϕ has dimension at least one. Then since $\mathrm{gl}(1, F)$ is one-dimensional, ϕ is onto. Then by Exercise 2.8a, $\phi(L') = \mathrm{gl}(1, F)' = 0$. □

Proposition 12.11 (Exericse 7.11)**.** *Let L be a Lie algebra over F such that $L' = L$. Then the only 1-dimensional representation of L is the trivial representation.*

Proof. Let $\phi : L \to \mathrm{gl}(1, F)$ be a representation. By part (a), $\phi(L') = 0$. Since $L = L'$, this implies $\phi(L) = \phi(L') = 0$. Thus ϕ is the trivial representation. □

Proposition 12.12 (Exercise 7.11)**.** *Let L be a Lie algebra, and let $\phi : L/L' \to \mathrm{gl}(V)$ be a representation. Thend define $\bar{\phi} : L \to \mathrm{gl}(V)$ by $\bar{\phi}(x) = \phi(x + L')$. We claim that $\bar{\phi}$ is a representation, and $\bar{\phi}(L') = 0$.*

Proof. Let $\pi : L \to L/L'$ be the canonical homomorphism given by $\pi(x) = x + L'$. Notice that we have defined $\bar{\phi}$ such that $\bar{\phi} = \phi\pi$, that is, the following diagram commutes.

$$\begin{tikzcd} L \arrow[r, "\pi"] \arrow[dr, "\bar{\phi}"'] & L/L' \arrow[d, "\phi"] \\ & \mathrm{gl}(V) \end{tikzcd}$$

Since π and ϕ are both homomorphisms, it follows that $\bar{\phi}$ is also a homomorphism, and thus it is a representation. Now we show that for $z \in L'$, $\bar{\phi}(z) = 0$.

$$\bar{\phi}(z) = \phi(z + L') = \phi(L') = 0$$

□

Proposition 12.13 (Exercise 7.11)**.** *Let L be a Lie algebra over $\mathbb{C}$, such that $L' \neq L$. Then L has infinitely many non-isomorphic 1-dimensional representations.*

Proof. Let $\mathrm{span}\{y_1, y_2, \ldots y_n\}$ be a basis for L' and extend this basis to a basis for L, so that $L = \mathrm{span}\{y_1, y_2, \ldots y_n, x_1, x_2, \ldots x_m\}$. Let $a \in \mathbb{C}$. We define a linear map $\tilde{a} : L \to \mathrm{gl}(\mathbb{C})$ on the basis elements of L as follows.

$$\tilde{a}(y_i)(c) = 0$$
$$\tilde{a}(x_i)(c) = ac$$

for all $c \in \mathbb{C}$. We claim that $\tilde{a}$ is a representation. By definition, $\tilde{a}$ is linear. We just need to check that $\tilde{a}$ preserves the bracket, that is,

$$\tilde{a}([w, z]) = [\tilde{a}(w), \tilde{a}(z)]$$

for all $w, z \in L$. Since $[w, z] \in L'$, we know that $[w, z] = \sum_{i=1}^{n} c_i y_i$ and thus $\tilde{a}([w, z])$ is the zero map. On the right hand side, the bracket is in $\mathrm{gl}(\mathbb{C})$, which is one dimensional, so it must be abelian. Thus both sides of this equation are the zero map for all $w, z \in L$, so they are equal. Thus $\tilde{a}$ is a representation.

We have shown that for each $a \in \mathbb{C}$, there is a representation $\tilde{a} : L \to \text{gl}(\mathbb{C})$. We claim that each $a \in \mathbb{C}$ gives a unique reprepresentation with respect to isomorphism. That is, we claim that if $a, b \in \mathbb{C}$ and $a \neq b$, then $\tilde{a}$ is not isomorphic to $\tilde{b}$.

Suppose there was an isomorphism $\theta : \mathbb{C} \to \mathbb{C}$. Since θ is a linear map, $\theta(c) = \lambda c$ for some $\lambda \in \mathbb{C}$. Then since θ is an isomorphism,

$$\theta(\tilde{a}(x)(c)) = \tilde{b}(x)\theta(c)$$

for all $x \in L, c \in \mathbb{C}$. In particular, this must hold for $x = x_1$.

$$\begin{aligned} \theta(\tilde{a}(x_1)(c)) &= \theta(ac) = \lambda ac \\ \tilde{b}(x_1)\theta(c) &= b\theta(c) = b\lambda c \\ \implies a\lambda c &= b\lambda c \end{aligned}$$

So if there is such an isomorphism θ, then $\lambda = 0$ or $a = b$. If $\lambda = 0$ then θ is not bijective, so the only possibility is $a = b$. Thus we have proven our claim that if $a \neq b$, then $\tilde{a}$ is not isomorphic to $\tilde{b}$. To summarize, for each $a \in \mathbb{C}$, there is a unique (up to isomorphism) representation $\tilde{a} : L \to \text{gl}(\mathbb{C})$, so there are uncountably infinitely many non-isomorphic 1-dimensional representations of L. ☐

Proposition 12.14 (Exercise 7.12i). *Let L be a Lie algebra over F and let V be an L-module. On the dual space V^*, define an action on L by*

$$(x \cdot \theta)(v) = -\theta(x \cdot v)$$

for $x \in L, v \in V, \theta \in V^*$. This action gives V^* the structure of an L-module.

Proof. We need to show that the conditions M1, M2, and M3 on page 55 hold. Let $x, y \in L, \theta, \psi \in V^*, a, b \in F$, and $v \in V$.

$$\begin{aligned} ((ax + by) \cdot \theta)(v) &= -\theta((ax + by) \cdot v) \\ &= -\theta(a(x \cdot v) + b(y \cdot v)) \\ &= -a\theta(x \cdot v) - b\theta(y \cdot v) \\ &= a(x \cdot \theta)(v) + b(y \cdot \theta)(v) \\ &= ((ax \cdot \theta) + b(y \cdot \theta))(v) \\ \implies (ax + by) \cdot \theta &= a(x \cdot \theta) + b(y \cdot \theta) \end{aligned}$$

Thus condition M1 is satisfied.

$$\begin{aligned} x \cdot (a\theta + b\psi)(v) &= -(a\theta + b\psi)(x \cdot v) \\ &= -a\theta(x \cdot v) - b\psi(x \cdot v) \\ &= a(x \cdot \theta)(v) + b(x \cdot \psi)(v) \\ &= (a(x \cdot \theta) + b(x \cdot \psi))(v) \\ \implies x \cdot (a\theta + b\psi) &= a(x \cdot \theta) + b(x \cdot \psi) \end{aligned}$$

Thus condition M2 is satisfied.

$$\begin{aligned}
([x,y]\cdot\theta)(v) &= -\theta([x,y]\cdot v) \\
&= -\theta(x\cdot(y\cdot v) - y\cdot(x\cdot v)) \\
&= -\theta(x\cdot(y\cdot v)\theta(y\cdot(x\cdot v)) \\
&= (x\cdot\theta)(y\cdot v) - (y\cdot\theta)(x\cdot v) \\
&= -(y\cdot(x\cdot\theta))(v) + (x\cdot(y\cdot\theta))(v) \\
&= -(y\cdot(x\cdot\theta) + x\cdot(y\cdot\theta))(v) \\
&= (x\cdot(y\cdot\theta) - y\cdot(x\cdot\theta))(v) \\
\implies ([x,y])\cdot\theta &= (x\cdot(y\cdot\theta)) - (y\cdot(x\cdot\theta))
\end{aligned}$$

Thus M3 is satisfied. Thus V^* is an L-module with this action. ☐

Lemma 12.15 (for Exercise 7.12i)**.** *Let V be a Lie module for L, and fix a basis β of V. If*

$$[v]^T[x]^T[w] = -[v][x][w]$$

for all $x \in L, v, w \in V$, then $[x]^T = -[x]$ for $x \in L$.

Proof. Let $\beta = \{e_1, e_2, \ldots\}$. Then since we have this equation for all $v, w \in V$, let $v = e_i$ and $w = e_j$. Then multiplying $[x]^T$ by $[e_i]^T$ on the left picks off the ith row of $[x]^T$, and multiplying the resulting row vector on the right by $[e_j]$ picks off the jth entry, so

$$[v]^T[x]^T[w] = [e_i]^T[x]^T[e_j] = ([x]^T)_{ij}$$

Likewise,

$$[v]^T(-[x])[w] = [e_i]^T(-[x])[e_j] = (-[x])_{ij}$$

Thus $([x]^T)_{ij} = (-[x])_{ij}$ for each $1 \le i, j \le \dim V$, so $[x]^T = -[x]$. ☐

Proposition 12.16 (Exercise 7.12i)**.** *Let V be an module for the Lie algebra L. Let $\{e_1, e_2, \ldots\}$ be a basis of V. Define a linear map $\psi : V \to V^*$ by $\psi(e_i) = \theta^i$ where $\theta^i(e_j) = \delta_{ij}$. Then ψ is an isomorphism if and only if the matrices representing the action of L in the basis β are skew-symmetric.*

Proof. Note that in terms of the basis β, we think of e_i as being a column vector. Then θ^i is a row vector, and $[\theta^i] = [e_i]^T$. More generally, $[\psi(v)] = [v]^T$.

Suppose ψ is an isomorphism. Then for $x \in L, v, w \in V$, we have $\psi(x\cdot v)(w) = (x \cdot \psi(v))(w) = -\psi(v)(x\cdot w)$. Now we think of this equation in terms of matrix representations, so we think of $[v], [w]$ as column vectors.

$$\begin{aligned}
[\psi(x\cdot v)] &= ([x][v])^T = [v]^T[x]^T \\
[\psi(x\cdot v)(w)] &= [\psi(x\cdot v)][w] = [v]^T[x]^T[w] \\
[-\psi(v)(x\cdot w)] &= -[v]^T[x]w \\
\implies [v]^T[x]^T[w] &= [v]^T(-[x])[w]
\end{aligned}$$

for all $x \in L, v, w \in V$. Then by the lemma, $[x]^T = -[x]$, so all the matrices representing the action of L are skew-symmetric.

Now suppose that the matrices $[x]$ for $x \in L$ are skew symmetric (with respect to the basis $\beta = \{e_1, e_2, \ldots\}$). Then we claim ψ is an isomorphism. We need to show that $\psi(x \cdot v)(w) = (x \cdot \psi(v))(w)$ for $x \in L, v, w \in V$.

$$\begin{aligned}
[\psi(x \cdot v)(w)] &= [\psi([x][v])(w)] \\
&= ([x][v])^T [w] \\
&= [v]^T [x]^T [w] \\
&= -[v]^T [x][w] \\
&= -[\psi(v)][x][w] \\
&= [-\psi(v)(x \cdot w)] \\
&= [(x \cdot \psi(v))(w)]
\end{aligned}$$

Thus $\psi(x \cdot v)(w)$ and $(x \cdot \psi(v))(w)$ have the same matrix representation (with respect to β), so they are equal. □

Proposition 12.17 (Exercise 7.12ii)**.** *Let L be a Lie algebra over F, and let V, W be L-modules. Define an action*

$$\begin{gathered}
L \times \operatorname{Hom}(V, W) \to \operatorname{Hom}(V, w) \\
(x \cdot \theta)(v) = x \cdot (\theta(v)) - \theta(x \cdot v)
\end{gathered}$$

for $x \in L, v \in V, \theta \in \operatorname{Hom}(V, W)$. This action gives $\operatorname{Hom}(V, W)$ the structure of an L-module.

Proof. We must show that the equations M1, M2, and M3 hold. Let $a, b \in F, x, y \in L, v \in V$, and $\theta, \psi \in \operatorname{Hom}(V, W)$.

$$\begin{aligned}
((ax + by) \cdot \theta)(v) &= (ax + by) \cdot (\theta(v)) - \theta((ax + by) \cdot v) \\
&= a(x \cdot (\theta(v))) + b(y \cdot (\theta(v))) - \theta(a(x \cdot v) + b(y \cdot v)) \\
&= a(x \cdot (\theta(v)) - \theta(x \cdot v)) + b(y \cdot (\theta(v)) - \theta(y \cdot v)) \\
&= a(x \cdot \theta)(v) - b(y \cdot \theta)(v) \\
&= (a(x \cdot \theta) - b(y \cdot \theta))(v) \\
\implies (ax + by) \cdot \theta &= a(x \cdot \theta) - b(y \cdot \theta)
\end{aligned}$$

Thus M1 holds.

$$\begin{aligned}
(a\theta + b\psi)(v) &= x \cdot ((a\theta + b\psi)(v)) - (a\theta + b\psi)(x \cdot v) \\
&= x \cdot (a\theta(v) + b\psi(v)) - (a\theta(x \cdot v)) - b\psi(x \cdot v) \\
&= a(x \cdot (\theta(v))) + b(x \cdot (\psi(v))) - a\theta(x \cdot v) - b\psi(x \cdot v) \\
&= a(x \cdot (\theta(v)) - \theta(x \cdot v)) + b(x \cdot (\psi(v)) - \psi(x \cdot v)) \\
&= a(x \cdot \theta)(v) + b(x \cdot \psi)(v) \\
&= (a(x \cdot (\theta) + b(x \cdot \psi))(v) \\
\implies x \cdot (a\theta + b\psi) &= a(x \cdot \theta) + b(x \cdot \psi)
\end{aligned}$$

Thus M2 holds. To show M3 holds, we must show that $[x, y] \cdot \theta = x \cdot (y \cdot \theta) - y \cdot (x \cdot \theta)$ as maps, so we need to show that for $v \in V$ these maps act on v in the same way. First we compute $([x, y] \cdot \theta)(v)$.

$$\begin{aligned}([x, y] \cdot \theta)(v) &= [x, y] \cdot (\theta(v)) - \theta([x, y] \cdot v)\\ &= x \cdot (y \cdot (\theta(v))) - y \cdot (x \cdot (\theta(v))) - \theta(x \cdot (y \cdot v) - y \cdot (x \cdot v))\\ &= x \cdot (y \cdot (\theta(v))) - y \cdot (x \cdot (\theta(v))) - \theta(x \cdot (y \cdot v)) + \theta(y \cdot (x \cdot v))\end{aligned}$$

Now we compute $(x \cdot (y \cdot \theta) - y \cdot (x \cdot \theta))(v)$. We compute the two terms separately after expanding.

$$\begin{aligned}(x \cdot (y \cdot \theta) - y \cdot (x \cdot \theta))(v) &= (x \cdot (y \cdot \theta))(v) - (y \cdot (x \cdot \theta))(v)\\ (x \cdot (y \cdot \theta))(v) &= x \cdot ((y \cdot \theta)(v)) - (y \cdot \theta)(x \cdot v)\\ &= x \cdot (y \cdot (\theta(v))) - x \cdot \theta(y \cdot v)) - y \cdot (\theta(x \cdot v)) + \theta(y \cdot (x \cdot v))\\ (y \cdot (x \cdot \theta))(v) &= y \cdot (x \cdot (\theta(v)) - y \cdot (\theta(x \cdot v)) - x \cdot (\theta(y \cdot v)) + \theta(x \cdot (y \cdot v))\end{aligned}$$

Now using the computations for the two terms, we get an expression for $(x \cdot (y \cdot \theta) - y \cdot (x \cdot \theta))(v)$ involving eight terms.

$$\begin{aligned}(x \cdot (y \cdot \theta) - y \cdot (x \cdot \theta))(v) = x \cdot (y \cdot (\theta(v))) - x \cdot \theta(y \cdot v)) - y \cdot (\theta(x \cdot v)) + \theta(y \cdot (x \cdot v))\\ + y \cdot (x \cdot (\theta(v)) - y \cdot (\theta(x \cdot v)) - x \cdot (\theta(y \cdot v)) + \theta(x \cdot (y \cdot v))\end{aligned}$$

Fortunately, two pairs of these terms cancel: we have a $-x \cdot (\theta(y \cdot v))$ term and a $x \cdot (\theta(y \cdot v))$ term, which cancel each other, and also the pair $-y \cdot (\theta(x \cdot v))$ and $y \cdot (\theta(x \cdot v))$. This leaves

$$\begin{aligned}(x \cdot (y \cdot \theta) - y \cdot (x \cdot \theta))(v) = x \cdot (y \cdot (\theta(v))) + \theta(y \cdot (x \cdot v))\\ + y \cdot (x \cdot (\theta(v)) + \theta(x \cdot (y \cdot v))\end{aligned}$$

and one can match up these terms one by one with the four terms in our expression for $([x, y] \cdot \theta)(v)$ computed earlier. Thus, we have shown that $\mathrm{Hom}(V, W)$ is an L-module with this action. □

Proposition 12.18 (Exercise 7.12ii). *Let V, W be L-modules and define an L-module structure on* $\mathrm{Hom}(V, W)$ *by*

$$(x \cdot \theta)(v) = x \cdot (\theta(v)) - \theta(x \cdot v)$$

for $x \in L, \theta \in \mathrm{Hom}(V, W)$, and $v \in V$. Then θ is an L-module homomorphism if and only if $x \cdot \theta = 0$ for all $x \in L$.

Proof. Suppose $x \cdot \theta = 0$ for all $x \in L$. Then for $v \in V$,

$$(x \cdot \theta)(v) = x \cdot (\theta(v)) - \theta(x \cdot v) = 0 \implies \theta(x \cdot v) = x \cdot (\theta(v))$$

thus θ is an L-module homomorphism. Now suppose that θ is an L-module homomorphism. Then for $x \in L, v \in V$,

$$\theta(x \cdot v) = x \cdot (\theta(v)) \implies x \cdot (\theta(v)) - \theta(x \cdot v) = (x \cdot \theta)(v) = 0 \implies x \cdot \theta = 0$$

□

13 Chapter 8 Exercises

Proposition 13.1 (Exercise 8.1)**.** *Let $\{e, f, h\}$ be the usual basis for $\mathrm{sl}(2, \mathbb{C})$ and let V_d be the vector space of homogenous polynomials in x and y of degree d. Let β be the usual basis for V_d.*

$$\beta = \{x^d, x^{d-1}y, \ldots y^d\}$$

Define a module structure by $\phi : \mathrm{sl}(2, \mathbb{C}) \to \mathrm{gl}(V_d)$ by

$$\phi(e) = x\frac{\partial}{\partial y} \qquad \phi(f) = y\frac{\partial}{\partial x} \qquad \phi(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

Then for any basis element $x^a y^b$, the submodule generated by $x^a y^b$ is all of V_d.

Proof. Let $W_k = \mathrm{span}\{x^k y^{d-k}\}$ for $k = 0, 1, \ldots d$. Then $V_d = W_0 \oplus W_1 \oplus \ldots \oplus W_d$. We compute the image of W_k under the action of $\phi(e)$ and $\phi(f)$.

$$\phi(e)W_k = \mathrm{span}\{x\frac{\partial}{\partial y}x^k y^{d-k}\} = \mathrm{span}\{(d-k)x^{k+1}y^{d-k-1}\} = W_{k+1}$$
$$\phi(f)W_k = \mathrm{span}\{y\frac{\partial}{\partial x}x^k y^{d-k}\} = \mathrm{span}\{kx^{k-1}y^{d-k+1}\} = W_{k-1}$$

The submodule generated by $x^a y^b$ contains W_a, and then by action of $\phi(e)$, it also contains $W_{a+1}, W_{a+2}, \ldots W_d$. By the actino of $\phi(f)$, it also contains $W_{a-1}, W_{a-2}, \ldots W_0$. Thus the submodule contains each W_k, so it contains all of V_d. □

Proposition 13.2 (Exercise 8.2i)**.** *Let $\psi : \mathrm{sl}(2, \mathbb{C}) \to \mathrm{gl}(\mathbb{C})$ be the trivial representation and let $\phi : \mathrm{sl}(2, \mathbb{C}) \to \mathrm{gl}(V_0)$ be the representation given on pages 67-68 of Erdmann and Wildon. Define $\theta : \mathbb{C} \to V_0$ by $\theta(x) = x$. Then θ is a Lie module isomorphism.*

Proof. V_0 is equal to $\mathbb{C}$ as a set, since V_0 is the set of constant polynomials over $\mathbb{C}$. Since θ is the identity on $\mathbb{C}$, it is a bijection and it is linear. We need to show that $\theta(\psi(x)v) = \phi(x)\theta(v)$ for all $v \in \mathbb{C}$ and each x in some basis of $\mathrm{sl}(2, \mathbb{C})$. We use the usual basis e, f, h. Since $\psi(x)v = 0$, we have $\theta(\psi(x)v) = 0$ for all $x \in \mathrm{sl}(2, \mathbb{C})$. Thus the LHS of our equation to show is always zero.

$$\phi(e)v = x\frac{\partial}{\partial y}v = 0$$
$$\phi(f)v = y\frac{\partial}{\partial x}v = 0$$
$$\phi(h)v = x\frac{\partial}{\partial x}v + y\frac{\partial}{\partial y}v = 0 + 0 = 0$$

Thus the RHS of our equation to show is also always zero. □

Proposition 13.3 (Exercise 8.2ii)**.** *Let $\psi : \mathrm{sl}(2, \mathbb{C}) \to \mathrm{gl}(\mathbb{C}^2)$ be the natural representation and let $\phi : \mathrm{sl}(2, \mathbb{C}) \to \mathrm{gl}(V_1)$ be the representation described by Erdmann and Wildon. Define $\theta : \mathbb{C}^2 \to V_1$ by*

$$\theta \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = v^1 x + v^2 y$$

Then θ is a Lie module isomorphism.

Proof. θ is clearly a bijection. We show that θ is linear.

$$\begin{aligned}\theta\left(\lambda\begin{pmatrix}v^1\\v^2\end{pmatrix}+\begin{pmatrix}w^1\\w^2\end{pmatrix}\right)&=\theta\begin{pmatrix}\lambda v^1+w^1\\\lambda v^2+w^2\end{pmatrix}\\&=(\lambda v^1+w^1)x+(\lambda v^2+w^2)y\\&=\theta\left(\lambda\begin{pmatrix}v^1\\v^2\end{pmatrix}\right)+\theta\begin{pmatrix}w^1\\w^2\end{pmatrix}\end{aligned}$$

Thus θ is linear. We need to show that $\theta(\psi(x)v)=\phi(x)\theta(v)$ for $v\in\mathbb{C}^2$ and all x in some basis of $\mathrm{sl}(2,\mathbb{C})$. We use the usual basis e,f,h.

$$\theta(\psi(e)v)=\theta\left(\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}v^1\\v^2\end{pmatrix}\right)=\theta\begin{pmatrix}v^2\\0\end{pmatrix}=v^2x=\phi(e)(v^1x+v^2y)=\phi(e)\theta(v)$$

$$\theta(\psi(f)v)=\theta\left(\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}v^1\\v^2\end{pmatrix}\right)=\theta\begin{pmatrix}0\\v^1\end{pmatrix}=v^1y=\phi(f)(v^1x+v^2y)=\phi(f)\theta(v)$$

$$\theta(\psi(h)v)=\theta\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}v^1\\v^2\end{pmatrix}\right)=\theta\begin{pmatrix}v^1\\-v^2\end{pmatrix}=v^1x-v^2y=\phi(h)(v^1x+v^2y)=\phi(h)\theta(v)$$

□

Proposition 13.4 (Exercise 8.2iii)**.** *The linear map* $\theta:(2,\mathbb{C})\to\mathbb{V}_{\not\in}$ *given by*

$$\theta(e)=x^2\qquad\theta(h)=-2xy\qquad\theta(f)=-y^2$$

is an isomorphism of Lie modules.

Proof. θ is a linear bijection by definition. We need to show that $\theta(\operatorname{ad}x(y))=\phi(x)\theta(y)$ for $x,y\in\{e,f,h\}$.

$$\begin{aligned}\theta([e,f])&=\theta(h)=-2xy=x\frac{\partial}{\partial y}(-y^2)=\phi(e)\theta(f)\\\theta([e,h])&=\theta(-2e)=-2\theta(e)=-2x^2=x\frac{\partial}{\partial y}(-2xy)=\phi(e)\theta(h)\\\theta([f,e])&=\theta(-h)=-\theta(h)=2xy=y\frac{\partial}{\partial x}x^2=\phi(f)\theta(e)\\\theta([f,h])&=\theta(2f)=-2y^2=y\frac{\partial}{\partial x}(-2xy)=\phi(f)\theta(h)\\\theta([h,e])&=\theta(2e)=2x^2=\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)x^2=\phi(h)\theta(e)\\\theta([h,f])&=\theta(-2f)=2y^2=\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)(-y^2)=\phi(h)\theta(f)\end{aligned}$$

□

Proposition 13.5 (Exercise 8.3)**.** *The subalgebra of* $\mathrm{sl}(2,\mathbb{C})$ *consisiting of matrices of the form*

$$\begin{pmatrix}*&*&0\\*&*&0\\0&0&0\end{pmatrix}$$

is isomorphic to $\mathrm{sl}(2,\mathbb{C})$.

Proof. Let A be the subalgebra in question. In terms of 2×2 unit matrices, the usual basis e, f, h for $\mathrm{sl}(2, \mathbb{C})$ is the matrices $\{e, f, h\} = \{e_{12}, e_{21}, e_{11} - e_{22}\}$. One can see that the 3×3 unit matrices $\{e_{12}, e_{21}, e_{11} - e_{22}\}$ are a basis for A. We define a linear map $\phi : \mathrm{sl}(2, \mathbb{C}) \to A$ by

$$\phi(e_{12}) = e_{12} \qquad \phi(e_{21}) = e_{21} \qquad \phi(e_{11} - e_{22}) = e_{11} - e_{22}$$

where e_{ij} may refer to either a 2×2 or 3×3 matrix depending on context. and claim that ϕ is a Lie algebra isomorphism. Since it maps basis to basis, it is a bijection and linear. As shown in chapter 1, the rule for the bracket of unit matrices is

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}$$

So clearly, the brackets in $\mathrm{sl}(2, \mathbb{C})$ and A are preserved by ϕ since ϕ simply changes the interpretation of unit matrix from 2×2 to 3×3. $\square$

Proposition 13.6 (Exercise 8.3)**.** *Because of the previous proposition, we can view* $\mathrm{sl}(3, \mathbb{C})$ *as a module for* $\mathrm{sl}(2, \mathbb{C})$ *with the action* $x \cdot y = [\phi(x), y]$ *where* $\phi : \mathrm{sl}(2, \mathbb{C}) \to \mathrm{sl}(3, \mathbb{C})$ *is the map*

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ c & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

As an $\mathrm{sl}(2, \mathbb{C})$ *module,* $\mathrm{sl}(3, \mathbb{C}) \cong V_2 \oplus V_1 \oplus V_1 \oplus V_0$.

Proof. Let $\{e, f, h\}$ be the usual basis for $\mathrm{sl}(2, \mathbb{C})$. In terms of the matrix units, $e = e_{12}, f = e_{21}$ and $h = e_{11} - e_{22}$. Conveniently, $\phi(e_{ij}) = e_{ij}$, where on the LHS we view e_{ij} as a 2×2 matrix and on the RHS we view e_{ij} as a 3×3 matrix. Consider $e_{13} \in \mathrm{sl}(3, \mathbb{C})$. Using the rule $[e_{ij}, e_{kl}] = \delta_{ji} e_{il} - \delta_{il} e_{kj}$, we compute

$$\begin{aligned} h \cdot e_{13} &= [h, e_{13}] = [e_{11} - e_{22}, e_{13}] = [e_{11}, e_{13}] - [e_{22}, e_{13}] = e_{13} \\ e \cdot e_{13} &= [e_{12}, e_{13}] = 0 \end{aligned}$$

Thus by Corollary 8.6, the submodule generated by e_{13} is isomorphic to V_1. Since $f \cdot e_{13} = e_{23}$, this submodule contains the linearly independent vectors $\{e_{13}, e_{23}\}$. Since V_1 is 2-dimensional, we conclude that $< e_{13} >= \mathrm{span}\{e_{13}, e_{23}\}$. Now consider e_{31}. We compute

$$\begin{aligned} h \cdot e_{32} &= e_{32} \\ e \cdot e_{32} &= 0 \\ f \cdot e_{32} &= -e_{31} \end{aligned}$$

So as before, we use Corollary 8.6 to conclude that $< e_{32} >\cong V_1$, and $< e_{32} >= \mathrm{span}\{e_{32}, e_{31}\}$. Now consider e_{12}. We compute

$$\begin{aligned} h \cdot e_{12} &= 2e_{12} \\ e \cdot e_{12} &= 0 \\ f \cdot e_{12} &= -(e_{11} - e_{22}) \\ f \cdot (f \cdot e) = f \cdot (-h) &= -[f, h] = 2f = 2e_{21} \end{aligned}$$

Thus by Corollary 8.6, $< e_{12} > \cong V_2$ and $< e_{12} > = \operatorname{span}\{e_{12}, e_{21}, e_{11} - e_{22}\}$. Finally, we compute

$$\begin{aligned} h \cdot (e_{11} + e_{22} - 2e_{33}) = 0 \\ e \cdot (e_{11} + e_{22} - 2e_{33}) = 0 \end{aligned}$$

so by Corollary 8.6, $< e_{11} + e_{22} - 2e_{33} > \cong V_0$. Putting all of this together, we have

$$\begin{aligned} \mathrm{sl}(3, \mathbb{C}) &= \operatorname{span}\{e_{12}, e_{21}, e_{11} - e_{22}, e_{32}, e_{31}, e_{23}, e_{13}, e_{11} + e_{22} - 2e_{33}\} \\ &= < e_{12} > \oplus < e_{32} > \oplus < e_{13} > \oplus < e_{11} + e_{22} - 2e_{33} > \\ &\cong V_3 \oplus V_2 \oplus V_2 \oplus V_0 \end{aligned}$$

□

Lemma 13.7 (for Exercise 8.4). *Consider the* $\mathrm{sl}(2, \mathbb{C})$ *module* V_d *with* $d \in \mathbb{N} \cup \{0\}$. *Let* $W_r = \{v \in V_d : h \cdot v = rv\}$. *Then if* d *is even,* $\dim W_0 = 1$ *and* $\dim W_1 = 0$. *If* d *is odd, then* $\dim W_0 = 0$ *and* $\dim W_1 = 1$.

Proof. As shown on page 68 of Erdmann and Wildon, for a basis element $x^a y^{d-a}$ of V_d,

$$h \cdot x^a y^{d-a} = (a - (d - a))x^a y^{d-a} = (2a - d)x^a y^{d-a}$$

Suppose d is even. Then $a = d/2$ is an integer, so

$$h \cdot x^{d/2} y^{d/2} = (d/2 - d/2)x^{d/2} y^{d/2} = 0$$

but for no other basis element is $h \cdot x^a y^{d-a}$ equal to zero, so $W_0 = \operatorname{span}\{x^{d/2} y^{d/2}\}$. Since d is even, $2a - d$ is also even, so $2a - d \neq 1$, so $W_1 = \{0\}$, so $\dim W_1 = 0$.

Suppose d is odd. Then $2a - d$ is odd, so $2a - d \neq 0$, so $W_0 = \{0\}$. But there is precisely one $a = (d + 1)/2$ such that $2a - d = 1$, so there is one basis element $x^{(d+1)/2} y^{(d-1)/2}$ such that

$$h \cdot x^{(d+1)/2} y^{(d-1)/2} = ((d + 1)/2 - (d - 1)/2)x^{(d+1)/2} y^{(d-1)/2} = x^{(d+1)/2} y^{(d-1)/2}$$

so $\dim W_1 = 1$. □

Corollary 13.8 (to previous lemma, for 8.4). *Let* V_d *be the* $\mathrm{sl}(2, \mathbb{C})$ *module defined by Erdmann and Wildon. Then* $\dim W_0 + \dim W_1 = 1$.

Proof. If d is even, $\dim W_0 + \dim W_1 = 1 + 0 = 1$. If d is odd, then $\dim W_0 + \dim W_1 = 0 + 1 = 1$. □

Proposition 13.9 (Exercise 8.4). *Let* V *be a finite-dimensional* $\mathrm{sl}(2, \mathbb{C})$ *module. Let* $W_r = \{v \in V : h \cdot v = rv\}$. *Then if* V *is a direct sum of* k *irreducible modules,* $k = \dim W_0 + \dim W_1$.

Proof. By Weyl's Theorem and Theorem 8.5 of Erdmann and Wildon, we can write V as a direct sum of k irreducible modules, each of which is isomorphic to some V_d.

$$V = V_{d_1} \oplus V_{d_2} \oplus \ldots \oplus V_{d_k}$$

Then we define $W_{r_i} = \{v \in V_{d_i} : h \cdot v = rv\}$. Then

$$W_0 = \{v \in V : h \cdot v = 0\} = \bigoplus_{i=1}^{k} W_{0_i}$$
$$W_1 = \{v \in V : h \cdot v = 0\} = \bigoplus_{i=1}^{k} W_{1_i}$$

By the previous corollary, $\dim W_{0_i} \oplus W_{1_i} = 1$, because the dimension of a direct sum is the sum of the dimensions. Note also that in the following computation, we use the fact that the dimesion of a sum is not changed by changing the order of the summands.

$$\dim W_0 + \dim W_1 = \dim(W_0 \oplus W_1) = \dim\left(\bigoplus_{i=1}^{k} W_{0_i} \oplus \bigoplus_{i=1}^{k} W_{1_i}\right) = \dim \bigoplus_{i=1}^{k}(W_{0_i} \oplus W_{1_i}) = k$$

□

Lemma 13.10 (for Exercise 8.6i)**.** *Let V be an $\mathrm{sl}(2,\mathbb{C})$ module. Then for $v \in V$,*

$$fe \cdot v = (ef - h) \cdot v \tag{13.1}$$
$$ef \cdot v = (fe + h) \cdot v \tag{13.2}$$
$$he \cdot v = (eh + 2e) \cdot v \tag{13.3}$$
$$eh \cdot v = (he - 2e) \cdot v \tag{13.4}$$
$$hf \cdot v = (fh - 2f) \cdot v \tag{13.5}$$
$$fh \cdot v = (hf + 2f) \cdot v \tag{13.6}$$
$$\frac{1}{2}heh \cdot v = (\frac{1}{2}eh^2 + eh) \cdot v \tag{13.7}$$
$$\frac{1}{2}heh \cdot v = (\frac{1}{2}h^2e - he) \cdot v \tag{13.8}$$
$$\frac{1}{2}hfh \cdot v = (\frac{1}{2}fh^2 - fh) \cdot v \tag{13.9}$$
$$\frac{1}{2}hfh \cdot v = (\frac{1}{2}h^2f + hf) \cdot v \tag{13.10}$$

Proof. Proof of 0.1:

$$h \cdot v = [e, f] \cdot v = fe \cdot v - ef \cdot v \implies fe \cdot v = (ef - h) \cdot v$$

Proof of 0.2:

$$h \cdot v = (ef - fe) \cdot v \implies ef \cdot v = (fe + h) \cdot v$$

Proof of 0.3 and 0.4:

$$\begin{aligned}[e,h] \cdot v = (eh - he) \cdot v &\implies -2e \cdot v = (eh - he) \cot v \\ &\implies he \cdot v = (eh + 2e) \cdot v \\ &\implies eh \cdot v = (he - we) \cdot v\end{aligned}$$

Proof of 0.5 and 0.6:

$$\begin{aligned}[f,h]\cdot v = (fh-hf)\cdot v &\implies 2f\cdot v = (fh-hf)\cdot v\\ &\implies hf\cdot v = (fh-2f)\cdot v\\ &\implies fh\cdot v = (hf+2f)\cdot v\end{aligned}$$

Proof of 0.7:

$$\begin{aligned}[h,e]\cdot h\cdot v = (he-eh)h\cdot v &= (heh-eh^2)\cdot v\\ 2eh\cdot v &= (heh-eh^2)\cdot v\\ \frac{1}{2}heh\cdot v &= (eh+\frac{1}{2}eh^2)\cdot v\end{aligned}$$

Proof of 0.8:

$$\begin{aligned}h\cdot[h,e]\cdot v = h(he-eh)\cdot v &= (h^2e-heh)\cdot v\\ 2he\cdot v &= (h^2e-heh)\cdot v\\ heh\cdot v &= (h^2e-2he)\cdot v\\ \frac{1}{2}heh\cdot v &= (\frac{1}{2}h^2e-he)\cdot v\end{aligned}$$

Proof of 0.9:

$$\begin{aligned}[h,f]\cdot h = (hf-fh)\cdot h\cdot v &= (hfh-fh^2)\cdot v\\ -2fh\cdot v &= (hfh-fh^2)\cdot v\\ hfh\cdot v &= (fh^2-2fh)\cdot v\\ \frac{1}{2}hfh\cdot v &= (\frac{1}{2}fh^2-fh)\cdot v\end{aligned}$$

Proof of 0.10:

$$\begin{aligned}h\cdot[h,f]\cdot v = h(hf-fh)\cdot v &= (h^2f-hfh)\cdot v\\ -2hf\cdot v &= (h^2f-hfh)\cdot v\\ \frac{1}{2}\cdot v &= (\frac{1}{2}+hf)\cdot v\end{aligned}$$

□

Proposition 13.11 (Exercise 8.6.i)**.** *Let M be a finite-dimensional* $\mathrm{sl}(2,\mathbb{C})$ *module, and define $c: M\to M$ by*

$$c(v) = \left(ef+fe+\frac{1}{2}h^2\right)\cdot v$$

Then c is a homomorphism of $\mathrm{sl}(2,\mathbb{C})$ *modules.*

Proof. To show: We must show that c commutes with the actions of e, f, and h. First we show that c commutes with c.

$$\begin{aligned}
ef \cdot v &= (h + fe) \cdot v \\
e^2 f \cdot v &= (eh + efe) \cdot v \\
(e^2 f + efe + \frac{1}{2} eh^{)} \cdot v &= (eh + 2efe + \frac{1}{2} eh^2) \cdot v \\
e \cdot c(v) &= (2efe) + \frac{1}{2} heh) \cdot v \\
fe \cdot v &= (ef - h) \cdot v \\
fe^2 \cdot v &= (efe - he) \cdot v \\
(efe + fe^2 + \frac{1}{2} h^2 e) \cdot v &= (2efe - he + \frac{1}{2} h^2 e) \cdot v \\
c(e \cdot v) &= (2efe + \frac{1}{2} heh) \cdot v
\end{aligned}$$

Thus $e \cdot c(v) = c(e \cdot v)$. Now we show that c commutes with the action of f.

$$\begin{aligned}
ef \cdot v &= (h + fe) \cdot v \\
ef^2 \cdot v &= (hf + fef) \cdot v \\
(ef^2 + fef + \frac{1}{2} h^2 f) \cdot v &= (hf + 2fef + \frac{1}{2} h^2 f) \cdot v \\
c(f \cdot v) &= (2fef + \frac{1}{2} hfh) \cdot v \\
fe \cdot v &= (ef - h) \cdot v \\
f^2 e \cdot v &= (fef - fh) \cdot v \\
(f^2 e + fef + \frac{1}{2} fh^2) \cdot v &= (2fef - fh + \frac{1}{2} fh^2) \cdot v \\
f \cdot c(v) &= (2fef) + \frac{1}{2} hfh) \cdot v
\end{aligned}$$

Thus $c(f \cdot v) = f \cdot c(v)$. Finally, we show that c commutes with the action of h.

$$\begin{aligned}
hef \cdot v &= (ehf - 2ef) \cdot v \\
hfe \cdot v &= (fhe - 2fe) \cdot v \\
(hef + hfe) \cdot v &= (ehf + fhe + 2ef - 2fe) \cdot v \\
efh \cdot v &= (ehf + 2ef) \cdot v \\
feh \cdot v &= (fhe - 2fe) \cdot v \\
(efh + feh) \cdot v &= (ehf + fhe + 2ef - 2fe) \cdot v
\end{aligned}$$

Thus

$$c(h \cdot v) = (efh + feh + \frac{1}{2} h^3) \cdot v = (hef + hfe + \frac{1}{2} h^3) \cdot v = h \cdot c(v)$$

Thus c is an $\mathrm{sl}(2, \mathbb{C})$ module homomorphism. $\square$

Proposition 13.12 (Exercise 8.6ii). *Let V_d be an irreducible* $\mathrm{sl}(2,\mathbb{C})$ *module. Let $c : V_d \to V_d$ be the Casimir operator, which is defined by*

$$c(v) = (ef + fe + 1/2h^2) \cdot v$$

We have shown that c is a homomorphism. We claim that $c(v) = 1/2d(d+2)v$.

Proof. By Lemma 8.4, V_d contains a w such that $h \cdot x = \lambda w$ and $e \cdot w = 0$. By Corollary 8.6, $\lambda = d$. Then

$$c(w) = (ef + fe + 1/2h^2) \cdot w = ef \cdot w + fe \cdot w + 1/2h^2 \cdot w = dw + 0 + 1/2d^2 w$$

The last equality uses Exercise 8.5 to evaluate $ef \cdot w$.

$$c(w) = (1/2d + 1)dw = 1/2(d+2)dw$$

By Schur's Lemma, since c is an $\mathrm{sl}(2,\mathbb{C})$ module homomorphism from V_d to itself, c must be a scalar multiple of the identity transformation. Thus since $c(w) = 1/2(d+2)dw$ for some $w \in V_d$, it follows that $c(v) = 1/2(d+2)dv$ for all $v \in V_d$. □

Proposition 13.13 (Exercise 8.6iii). *Let M be a finite dimensional* $\mathrm{sl}(2,\mathbb{C})$ *module and let $c : M \to M$ be the Casimir operator. If*

$$M = \bigoplus_{i=1}^{r} \ker(c - \lambda_i I)^{m_i}$$

is the primary decomposition of M, then each $\ker(c - \lambda_i I)^{m_i}$ *is an* $\mathrm{sl}(2,\mathbb{C})$ *module.*

Proof. To show: For $v \in V_i = \ker(c - \lambda_i I)^{m_i}$, $e \cdot v, f \cdot v, h \cdot v \in V_i$. Let $v \in \ker(c - \lambda_i I)^{m_i}$. Note that $(c - \lambda_i I)^{m_k}$ is a polynomial in c. By part (i), e, f, h commute with c, so they commute with any polynomial in c, so

$$(c - \lambda_i I)^{m_i}(e \cdot v) = e \cdot (c - \lambda_i I)^{m_i}(v) = 0$$

(This is zero because $v \in \ker(c - \lambda_i I)^{m_i}$.) Likewise for f and h, the algebra is nearly identical. Thus $\ker(c - \lambda_i I)^{m_i}$ is an $\mathrm{sl}(2,\mathbb{C})$ submodule. □

Proposition 13.14 (Exercise 8.6iv). *Suppose M is a finite-dimensional* $\mathrm{sl}(2,\mathbb{C})$ *module such that M has just one generalized eigenspace of the Casimir operator c, that is, suppose*

$$M = \ker(c - \lambda I)^m$$

for some $\lambda \in \mathbb{C}$ and suppose that some irreducible submodule of M is isomorphic to V_d. Then every irreducible submodule of M is isomorphic to V_d.

Proof. Let U be an irreducible submodule of M such that $U \cong V_d$. Acting on M, c has only one eigenvalue λ. By part (ii), c acts on $U \cong V_d$ as the scalar $1/2(d)(d+2)$, so $\lambda = 1/2(d)(d+2)$. Let $N \cong V_k$ be another irreducible submodule of M. Then c acts on N by the saclar $1/2(k)(k+2)$, so $\lambda = 1/2(k)(k+2)$. Then $1/2(d)(d+2) = 1/2(k)(k+2)$ so $k = d$. Thus $N \cong V_d$. □

Proposition 13.15 (Exercise 8.6v)**.** *Let M be an* $\mathrm{sl}(2,\mathbb{C})$ *module such that* $M = \ker(c-\lambda I)^m$ *and M has an irreducible submodule isomorphic to V_d. Let N be a submodule of M. Then any irreducible submodule of M/N is isomorphic to V_d.*

Proof. By part (iv), M is isomorphic to a finite direct sum of V_d's,

$$M \cong V_d \oplus V_d \oplus \ldots \oplus V_d$$

Since V_d is irreducible, if any N is a submodule of M, then N is also isomorphic to a finite direct sum of V_d's, with less than or fewer summands than M.

$$N \cong V_d \oplus \ldots \oplus V_d$$

Then

$$M \cong N \oplus (V_d \oplus \ldots \oplus V_d)$$

and thus M/N is isomorphic to the remaining V_d's after "subtracting" N.

$$M/N \cong V_d \oplus \ldots \oplus V_d$$

Thus every irreducible submodule of M/N is isomorphic to V_d. □

Proposition 13.16 (Exercise 8.7)**.** *Define* $\psi : \mathbb{R}^3_\wedge \to \mathrm{sl}(2,\mathbb{C})$ *to be the linear map defined by*

$$\psi(x) = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \qquad \psi(y) = \begin{pmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix} \qquad \psi(z) = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}$$

We have shown that ψ is a Lie algebra isomorphism. We claim that

$$\psi(x)^2 + \psi(y)^2 + \psi(y)^2 = -\frac{3}{4}I$$

where I is the 2×2 identity matrix. Furthermore, we can write $\psi(x), \psi(y)$, and $\psi(z)$ in terms of e, f, and h, and then see that the Casimir operator is represented by $3/2I$ (in this representation).

Proof.

$$\psi(x)^2 = \psi(y)^2 = \psi(z)^2 = -1/4I$$
$$\psi(x)^2 + \psi(y)^2 + \psi(z)^2 = -3/4I$$

$$\begin{aligned} \psi(x) &= 1/2e - 1/2f \\ \psi(y) &= -i/2e - i/2f \\ \psi(z) &= -i/2h \\ e &= \psi(x) + i\psi(y) \\ f &= -\psi(x) + i\psi(y) \\ h &= 2i\psi(z) \end{aligned}$$

$$
\begin{aligned}
ef + fe + 1/2h^2 &= (\psi(x) + i\psi(y)(-\psi(x) + i\psi(y)) \\
&\quad + (-\psi(x) + i\psi(y))(\psi(x) + i\psi(y) + 1/2(2i\psi(z))^2 \\
&= -\psi(x)^2 - \psi(y)\psi(x) + i\psi(x)\psi(y) - \psi(y)^2 \\
&\quad - \psi(x)^2 + i\psi(y)\psi(x) - i\psi(x)\psi(y) - \psi(y)^2 + 1/2(4)(-1)\psi(z)^2 \\
&= -2\psi(x)^2 - 2\psi(y)^2 - 2\psi(z)^2 \\
&= -2(-3/4)I \\
&= 3/2I
\end{aligned}
$$

□

14 Chapter 9 Exercises

Proposition 14.1 (Exercise 9.1)**.** *Let V be a vector space, and suppose $x \in \mathrm{gl}(V)$ has Jordan decomposition $x = d + n$. Then $\operatorname{ad} x : \mathrm{gl}(V) \to \mathrm{gl}(V)$ has Jordan decomposition $\operatorname{ad} x = \operatorname{ad} d + \operatorname{ad} n$.*

Proof. To show: $\operatorname{ad} x = \operatorname{ad} d + \operatorname{ad} n$, and $\operatorname{ad} d$ is diagonalisable, and $\operatorname{ad} n$ is nilpotent, and $\operatorname{ad} d \circ \operatorname{ad} n = \operatorname{ad} n \circ \operatorname{ad} d$. Because ad is linear, $\operatorname{ad} x = \operatorname{ad}(d + n) = \operatorname{ad} d + \operatorname{ad} n$. Since d is diagonalisable, by Exercise 1.17, $\operatorname{ad} d$ is diagonalisable. Since n is nilpotent, by Lemma 5.1, $\operatorname{ad} n$ is nilpotent. Finally, let $\phi \in \mathrm{gl}(V)$. Then

$$\begin{aligned}
\operatorname{ad} d \circ \operatorname{ad} n(\phi) &= [d, [n, \phi]] \\
&= [d, n\phi - \phi n] \\
&= dn\phi - d\phi n - n\phi d + \phi nd \\
&= nd\phi - n\phi d - d\phi n + \phi dn \\
&= [n, d\phi - \phi d] \\
&= [n, [d, \phi]] \\
&= \operatorname{ad} n \circ \operatorname{ad} d(\phi)
\end{aligned}$$

□

Lemma 14.2 (for Exercise 9.2)**.** *Let A, B be $n \times n$ matrices such that A is upper triangular and B is strictly upper triangular. Then $\operatorname{tr}(AB) = 0$.*

Proof. Let $A = (a_{ij})B = (b_{ij})$. Then

$$\begin{aligned}
(AB)_{ij} &= \sum_{k=1}^{n} a_{ik} b_{kj} \\
(AB)_{ii} &= \sum_{k=1}^{n} a_{ik} b_{ki} \\
\operatorname{tr}(AB) &= \sum_{m=1}^{n} \sum_{k=1}^{n} a_{km} b_{mk}
\end{aligned}$$

Since both A, B are upper triangular, $a_{mk} = 0$ for $m < k$ and $b_{mk} = 0$ for $k < m$. Thus in our sum for the trace, all terms are zero except perhaps those of the form $a_{kk} b_{kk}$.

$$\operatorname{tr}(AB) = \sum_{k=1}^{n} a_{kk} b_{kk}$$

Since B is strictly upper triangular, $b_{kk} = 0$ for each k. Thus $\operatorname{tr}(AB) = 0$. □

Proposition 14.3 (Exercise 9.2)**.** *Let V be a complex vector space and let L be a solvable Lie subalgebra of $\mathrm{gl}(V)$. Then there is a basis of V in which every element of L' is represented by a strictly upper triangular matrix. Consequently, $\operatorname{tr} xy = 0$ for $x \in L, y \in L'$.*

Proof. By Lie's Theorem, there is a basis of V in which every $x \in L$ is represented by an upper triangular matrix. If $x_1, x_2 \in L$ are upper triangular and $x_3 = [x_1, x_2] \in L'$, then x_3 is strictly upper triangular, since the bracket of upper triangular matrices is strictly upper triangular (see Exercise 4.5). Thus L' has a basis consisting of strictly upper triangular matrices.

Let $x \in L, y \in L'$. Then since x is upper triangular and y is strictly upper triangular, by the lemma, $\operatorname{tr} xy = 0$. □

Proposition 14.4 (Exercise 9.3)**.** *Let L be a Lie algebra and let I be an ideal of L. Then $I^\perp$ is an ideal of L.*

Proof. To show: for $b \in I^\perp, x \in L$, we have $[b, x] \in I^\perp$. Let $b \in I^\perp, a \in I, x \in L$. By definition,

$$I^\perp = \{b \in L : \kappa(b, a) = 0 \text{ for } a \in I\}$$

Since I is an ideal, $[x, a] \in I$. Then $\kappa(b, [x, a]) = 0$, and by associativity of κ (see page 80 of Erdmann and Wildon),

$$\kappa([b, x], a) = 0$$

Since $a \in I$ was arbitrary, this shows that $[b, x] \in I^\perp$. Thus $I^\perp$ is an ideal of L. □

Proposition 14.5 (Exercise 9.4i)**.** *The Killing form of* $\mathrm{sl}(2, \mathbb{C})$ *has the matrix*

$$\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

with respect to the usual basis e, f, h. It is non-degenerate.

Proof. Computations were done in Mathematica.

$$\begin{array}{lll} \kappa(e,e) = 0 & \kappa(f,e) = 4 & \kappa(h,e) = 0 \\ \kappa(e,f) = 4 & \kappa(f,f) = 0 & \kappa(h,f) = 0 \\ \kappa(e,h) = 0 & \kappa(f,h) = 0 & \kappa(h,h) = 8 \end{array}$$

For finite-dimensional vector space, a bilinear form is non-degenerate if and only if its matrix representation is invertible. This matrix clearly has nonzero determinant, so the form is non-degenerate. □

Proposition 14.6 (Exercise 9.4ii)**.** *The Killing form κ on* $\mathrm{gl}(2, \mathbb{C})$ *is degenerate.*

Proof. Take the identity matrix I and let $x \in \mathrm{gl}(2, \mathbb{C})$.

$$\operatorname{ad} I(x) = [I, x] = Ix - xI = x - x = 0$$

so $\kappa(I, x) = 0$ for $x \in \mathrm{gl}(2, \mathbb{C})$. Thus $\mathrm{gl}(2, \mathbb{C})^\perp \neq \{O\}$, so κ is degenerate. □

Proposition 14.7 (Exercise 9.5)**.** *Let L be a nilpotent Lie algebra over a field F. Then the Killing form κ on L is always zero, that is, for $x, y \in L$,*

$$\kappa(x, y) = \operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y) = 0$$

Proof. Let $x, y \in L$. Since L is nilpotent, $\operatorname{ad} x, \operatorname{ad} y$ are nilpotent maps (Theorem 6.3). By Theorem 6.1, since $\operatorname{ad} L$ is a Lie subalgebra of $\operatorname{gl}(L)$ in which every $\operatorname{ad} x$ is nilpotent, there is a basis of L in which everything in $\operatorname{ad} L$ is represented by a strictly upper triangular matrix. Then $\operatorname{ad} x \circ \operatorname{ad} y$ is also represented by a strictly upper triangular matrix, so

$$\operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y) = 0$$

□

(Exercise 9.6)
We compute the Killing form for the complex 3-dimensional Lie algebras discussed in chapter 3. The Heisenberg algebra is nilpotent, so it has a Killing form that is always equal to zero. The algebra considered in 3.2.4 is isomorphic to $\operatorname{sl}(2, \mathbb{C})$, which we have already computed the Killing form for.

The Lie algebra in section 3.2.2 is given by $L = \operatorname{span}\{x, y, z\}$ where $[x, y] = x, [x, z] = [y, z] = 0$. Then one can compute the matrices of $\operatorname{ad} x, \operatorname{ad} y, \operatorname{ad} z$:

$$[\operatorname{ad} x] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad [\operatorname{ad} y] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad [\operatorname{ad} z] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From this, clearly $\kappa(z, a) = 0$ for any $a \in L$. We still need to compute $\kappa(x, y), \kappa(x, x)$, and $\kappa(y, y)$. To do this, we compute the matrix products $[\operatorname{ad} x][\operatorname{ad} y], [\operatorname{ad} x]^2, [\operatorname{ad} y]^2$ and take the traces.

$$[\operatorname{ad} x][\operatorname{ad} y] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad [\operatorname{ad} x]^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad [\operatorname{ad} y]^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so we get $\kappa(x, x) = \kappa(x, y) = 0$ but $\kappa(y, y) = 1$. Thus to completely characterize the Killing form, we can write either

$$\kappa(a, b) = \begin{cases} 1 & \text{if } a = b = y \\ 0 & \text{otherwise} \end{cases} \qquad [\kappa] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we consider the Lie algebras discussed in sectin 3.2.3, beginning with case 2. In case 2, there is only one isomorphism class, which is the Lie algebra $L = \operatorname{span}\{x, y, z\}$ with $[x, y] = y, [x, z] = y + z, [y, z] = 0$. Then we compute the matrices of $\operatorname{ad} x, \operatorname{ad} y, \operatorname{ad} z$.

$$[\operatorname{ad} x] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad [\operatorname{ad} y] = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad [\operatorname{ad} z] = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0` \end{pmatrix}$$

And now we compute all of the necessary matrix products.

$$[\operatorname{ad} x][\operatorname{ad} y] = 0$$
$$[\operatorname{ad} y][\operatorname{ad} z] = 0$$
$$[\operatorname{ad} x][\operatorname{ad} z] = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$
$$[\operatorname{ad} x]^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$
$$[\operatorname{ad} y]^2 = 0$$
$$[\operatorname{ad} z]^2 = 0$$

So $\kappa(x,x) = 2$ but it is zero for everything else. Thus we completely characterize κ by

$$\kappa(a,b) = \begin{cases} 2 & \text{if } a = b = x \\ 0 & \text{otherwise} \end{cases} \qquad [\kappa] = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we consider the class of 3-dimensional Lie algebras considered in Case 1 of section 3.2.3. These are the algebras such that $L = \operatorname{span}\{x,y,z\}$ and $[x,y] = y, [y,z] = 0, [x,z] = \lambda z$ for some fixed $\lambda \in \mathbb{C}$. (As was shown in 3.2.4, we get a non-isomorphic Lie algebra for each $\lambda \in \mathbb{C}$ except that the Lie algebra with λ^{-1} is isomorphic.) We compute the matrics of $\operatorname{ad} x, \operatorname{ad} y, \operatorname{ad} z$.

$$[\operatorname{ad} x] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad [\operatorname{ad} y] = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad [\operatorname{ad} z] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\lambda & 0 & 0 \end{pmatrix}$$

Now we compute the needed matrix products.

$$[\operatorname{ad} x][\operatorname{ad} y] = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$[\operatorname{ad} y][\operatorname{ad} z] = 0$$
$$[\operatorname{ad} x][\operatorname{ad} z] = 0$$
$$[\operatorname{ad} x]^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}$$
$$[\operatorname{ad} y]^2 = 0$$
$$[\operatorname{ad} z]^2 = 0$$

So we can characterize the Killing form by

$$\kappa(a,b) = \begin{cases} 1+\lambda^2 & \text{if } a = b = x \\ 0 & \text{otherwise} \end{cases} \qquad [\kappa] = \begin{pmatrix} 1+\lambda^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This concludes our characterization of the Killing forms of the 3-dimensional complex Lie algebras.

Proposition 14.8 (Exercise 9.10)**.** *Let L be a complex Lie algebra and let β by a symmetric, bilinear, associative form on L. Define*

$$\theta : L \to L^* \qquad \theta(x)y = \beta(x, y)$$

Then θ is linear. (We call θ the linear map induced by β.)

Proof. First, we need to show that θ is well-defined, that is, that $\theta(x)$ does map into L^*. To do this, we need to show that $\theta(x)$ is a linear map. Let $a, b \in \mathbb{C}$ and $x, y, z \in L$.

$$\theta(x)(ay + bz) = \beta(x, ay + bz) = a\beta(x, y) + b\beta(x, z) = a\theta(x)y + b\theta(x)z$$

Thus $\theta(x)$ is linear. Now we can show that θ is linear. To do that we consider how $\theta(ax+by)$ acts on a given $z \in L$.

$$\theta(ax + by)z = \beta(ax + by)z = a\beta(x, z) + b\beta(y, z) = a\theta(x)z + b\theta(y)z$$

Thus

$$\theta(ax + by) = a\theta(x) + b\theta(y)$$

(The above is an equality of maps.) Thus θ is linear. □

Proposition 14.9 (Exercise 9.10)**.** *Let L be a complex Lie algebra and let β be a symmetric, associative bilinear form on L. Define $\theta : L \to L^*$ by $\theta(x)y = \beta(x, y)$. If β is non-degenerate, then L and L^* are isomorphic as L-modules.*

Proof. First we need to show that θ is an L-module homomorphism. To do this, we need to show that for $x, y \in L$, $\theta(x \cdot y) = x \cdot \theta(y)$.

(Recall that when we regard L as an L-module, the action is simply the bracket, that is, $x \cdot y = [x, y]$. When we regard L^* as an L-module, the action is given by $(x \cdot \psi)y = -\psi(x \cdot y)$ where $x, y \in L$ and $\psi \in L^*$.)

Let $x, y, z \in L$. We confirm that $\theta(x \cdot y) = x \cdot \theta(y)$ by looking at how each of the two maps acts on some $z \in L$.

$$\begin{aligned}\theta(x \cdot y)z = \theta([x, y])z = \beta([x, y], z) = \beta([x, [y, z]) = -\beta([x, [z, y]) \\ = -\beta([x, z], y) = -\beta(y, [x, z]) = -\theta(y)([x, z]) = (x \cdot \theta(y))z\end{aligned}$$

Thus we have the desired equality of maps,

$$\theta(x \cdot y) = x \cdot \theta(y)$$

Thus θ is an L-module homomorphism. This did not depend on the non-degeneracy of β, but now we show that if β is non-degenerate, then θ is an isomorphism. To do this, we show that the kernel of θ is $\{0\}$. We claim that $L^\perp = \ker\theta$.

$$\begin{aligned}L^\perp &= \{x \in L : \beta(x, y) = 0, \forall y \in L\} \\ &= \{x \in L : \theta(x)y = 0, \forall y \in L\} \\ &= \{x \in L : \theta(x) = 0\} \\ &= \ker\theta\end{aligned}$$

If β is non-degenerate, then $L^\perp = \{0\}$, so then $\ker\theta = \{0\}$ which implies that θ is an isomorphism. $\square$

Proposition 14.10 (Exercise 9.11)**.** *Let L be a simple Lie algebra over $\mathbb{C}$ with Killing form κ. Let β be a symmetric, associative, non-degenerate bilinear form on L. Then $\kappa = \lambda\beta$ for some $\lambda \neq 0$ with $\lambda \in \mathbb{C}$.*

Proof. By Exercise 9.10, κ and β induce L-module isomorphisms $\theta_\kappa, \theta_\beta : L \to L^*$. Then $\theta_\kappa\theta_\beta^{-1} : L \to L$ is an L-module isomorphism. Since L is simple, it is an irreducible L-module (example 7.9(2) on page 59). By Schur's Lemma, $\theta_\kappa\theta_\beta^{-1} = \lambda 1_L$ for some $\lambda \in \mathbb{C}$. Then

$$\kappa(x, y) = \theta_\kappa(x)y = \lambda\theta_\beta(x)y = \lambda\beta(x, y)$$

Thus $\kappa = \lambda\beta$. (Note that $\lambda \neq 0$ since κ, β are both non-degenerate.) $\square$

(Exercise 9.13) We give an example to show that the requirement of d and n commuting in the Jodran decomposition is necessary. Specifically, we give two matrices d and n such that d is diagonalisable, n is nilpotent, but d and n do not commute.

$$d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad x = d + n = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

Note that d is diagonal, so it is certainly diagonalisable, and n is nilpotent since $n^2 = 0$. However,

$$nd = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \qquad dn = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Thus $dn \neq nd$.

Proposition 14.11 (Exercise 9.14)**.** *Let L be a complex semisimple Lie algebra. Suppose $\phi : L \to \mathrm{gl}(V)$ is a faithful representation of L such that $\phi(x)$ is diagonalisable for some $x \in L$. Then x is a semisimple element of L and thus x acts diagonalisably in any representation of L.*

Proof. Let x be as described. By Theorem 9.15, x can be written uniquely as $x = d+n$ in an abstract Jordan decomposition. Then by Theorem 9.16, the Jordan decomposition of $\phi(x)$ is $\phi(x) = \phi(d) + \phi(n)$, where $\phi(d)$ is diagonalisable. By hypothesis $\phi(x)$ is diagonalisable. We claim that $\phi(x)$ and $\phi(d)$ commute, since

$$\begin{aligned}
\phi(x)\phi(d) &= (\phi(d) + \phi(n))\phi(d) \\
&= \phi(d)^2 + \phi(n)\phi(d) \\
&= \phi(d)^2 + \phi(d)\phi(n) \\
&= \phi(d)(\phi(d) + \phi(n)) \\
&= \phi(d)\phi(x)
\end{aligned}$$

since $\phi(d), \phi(n)$ commute by definition of Jordan decomposition. Thus by Lemma 16.7, $\phi(x)$ and $\phi(d)$ are simultaneously diagonalisable with respect to some basis β. Then since

$$[\phi(x)]_\beta = [\phi(d)]_\beta + [\phi(n)]_\beta$$

it follows that $\phi(n)]_\beta = 0$, so $\phi(n) = 0$. Since ϕ is one-to-one, $n = 0$, thus x is semisimple. Thus if $\theta : L \to \mathrm{gl}(V)$ is any representation of L,

$$\theta(x) = \theta(d + n) = \theta(d + 0) = \theta(d)$$

where $\theta(d)$ is diagonalisable, so $\theta(x)$ is diagonalisable. $\square$

15 Chapter 10 Exercises

Proposition 15.1 (Exercise 10.1)**.** *Let L be a finite-dimensional semisimple complex Lie algebra and let H be a Cartan subalgebra. Let $\alpha \in H^*$ with $\alpha \neq 0$. Let*

$$L_\alpha = \{x \in L : [h, x] = \alpha(h)x, \forall h \in H\}$$

Fix $x \in L_\alpha$. Then $\operatorname{ad} x$ is nilpotent.

Proof. We can decompose L as

$$L = H \oplus \bigoplus_{\beta \in \Phi} L_\beta$$

where Φ is finite. Then

$$(\operatorname{ad} x)^2(H) = \operatorname{ad} x(\operatorname{ad} x(H)) = \operatorname{ad} x(\{[x, h] : h \in H\}) \subset \operatorname{ad} x(\operatorname{span}\{x\}) = \{0\}$$

Thus $(\operatorname{ad} x)^2(H) = \{0\}$. We will also show that $\operatorname{ad} x$ acting on any L_β is nilpotent. Let $\beta \in \Phi$.

$$\operatorname{ad} x(L_\beta) = \{[x, y] : y \in L_\beta\} \subset [L_\alpha, L_\beta] \subset L_{\alpha+\beta}$$

by Lemma 10.1(i). Then

$$(\operatorname{ad} x)^2(L_\beta) \subset \operatorname{ad} x(L_{\alpha+\beta}) \subset [L_\alpha, L_{\alpha+\beta}] \subset L_{2\alpha+\beta}$$

and by a simple induction

$$(\operatorname{ad} x)^n(L_\beta) \subset L_{n\alpha+\beta}$$

There are infinitely many $L_{n\alpha+\beta}$, but only finitely many nonzero root spaces of L. Thus for some n, we have $L_{n\alpha+\beta} = \{0\}$, so

$$(\operatorname{ad} x)^n(L_\beta) \subset \{0\}$$

thus $\operatorname{ad} x$ restricted to L_β is nilpotent. We have shown that $\operatorname{ad} x$ is nilpotent on each summand of L, so $\operatorname{ad} x$ acts nilpotently on all of L. □

Proposition 15.2 (Exercise 10.2)**.** *Let $L = \mathrm{sl}(n, \mathbb{C})$ with $n \geq 2$, and let $H = \operatorname{span}\{h\}$ where $h = e_{11} - e_{22}$. Then L decomposes into weight spaces as*

$$L = L_0 \oplus L_\alpha \oplus L_\beta$$

where the weights $\alpha, \beta : H \to \mathbb{C}$ are defined by $\alpha(h) = 1$ and $\beta(h) = -1$ and the corresponding weight spaces are

$$\begin{aligned} L_0 &= \{x \in L : [h, x] = 0\} \\ L_\alpha &= \{x \in L : [h, x] = x\} \\ L_\beta &= \{x \in L : [h, x] = -x\} \end{aligned}$$

Proof. We compute $[h, e_{ij}]$.

$$[h, e_{ij}] = [e_{11}, e_{ij}] - [e_{22}, e_{ij}] = \delta_{1i}e_{1j} - \delta_{1j}e_{1i} + \delta_{2i}e_{2j} - \delta_{2j}e_{i2}$$

When $i, j \geq 3$, all of the Kronecker deltas become zero, so $[h, e_{ij}] = 0$ for $i, j \geq 3$. Somewhat surprisingly, when $i, j \leq 2$, terms cancel and we again get zero. Thus $[h, e_{ij}] = 0$ for $i, j \leq 2$. When $i \leq 2$ but $j \geq 3$, then $[h, e_{ij}] = \delta_{1i}e_{1j} + \delta_{2i}e_{2j} = e_{ij}$. Thus for $i \leq 2, j \geq 3$, $\text{span}\{e_{ij}\} \subset L_\alpha$. When $i \geq 3$ and $j \leq 2$, then $[h, e_{ij}] = -\delta_{1j}e_{1j} - \delta_{2j}e_{i2} = -e_{ij}$. Thus in this case, $\text{span}\{e_{ij}\} \subset L_\beta$. Thus we have allocated the entirety of the standard basis each to one of L_0, L_α, L_β. By definition, these weight spaces have trivial intersection, thus $L = L_0 \oplus L_\alpha \oplus L_\beta$. □

Proposition 15.3 (Exercise 10.3i)**.** *Let L be a semisimple complex finite-dimensional Lie algebra with Cartan subalgebra H. Let $\alpha : H \to \mathbb{C}$ be a root of L with $L_\alpha \neq \{0\}$. Define* $\text{sl}(\alpha) = \text{span}\{x, y, [x, y]\}$ *where $x \in L_\alpha, y \in L_{-\alpha}$, and $[x, y] \in H$. We know that* $\text{sl}(\alpha) \cong \text{sl}(2, \mathbb{C})$. *Then there is a basis $\{e_\alpha, f_\alpha, h_\alpha\}$ of* $\text{sl}(\alpha)$ *such that $e_\alpha \in L_\alpha, f_\alpha \in L_{-\alpha}, h_\alpha \in H$, and $\alpha(h_\alpha) = 2$.*

Proof. We have fixed x, y, h. We know that $\alpha(h) \neq 0$ as shown in the Lemma. Let $\lambda = 2/\alpha(h)$, then set $e_\alpha = x, f_\alpha = \lambda y, h_\alpha = \lambda h$. Then $e_\alpha \in L_\alpha, f_\alpha \in_{-\alpha}$, and $h_\alpha \in H$, and

$$\alpha(h_\alpha) = \alpha(\lambda h) = \lambda\alpha(h) = \frac{2}{\alpha(h)}\alpha(h) = 2$$

□

Proposition 15.4 (Exercise 10.3ii)**.** *Let $\alpha, e_\alpha, f_\alpha, h_\alpha$ be as above. Then $\theta : \text{sl}(\alpha) \to \text{sl}(2, \mathbb{C})$ defined by*

$$\theta(e_\alpha) = e \qquad \theta(f_\alpha) = f \qquad \theta(h_\alpha) = h$$

is a Lie algebra isomorphism.

Proof. θ is a linear isomorphism of vector spaces because it maps a basis to a basis. We need to show it preserves the bracket.

$$\begin{aligned}
\theta([e_\alpha, f_\alpha]) &= \theta(h_\alpha) = h = [e, f] = [\theta(e_\alpha), \theta(f_\alpha)] \\
\theta([h_\alpha, e_\alpha]) &= \theta(\alpha(h_\alpha)e_\alpha) = 2\theta(e_\alpha) = 2e = [h, e] = [\theta(h_\alpha), \theta(e_\alpha)] \\
\theta([h_\alpha, f_\alpha]) &= \theta(-2f_\alpha) = -2f_\alpha = [h, f] = [\theta(h_\alpha), \theta(f_\alpha)]
\end{aligned}$$

□

Proposition 15.5 (Exercise 10.5)**.** *Let L be a complex, finite-dimensional, semisimple Lie algebra with Cartan subalgebra H. Let $|\Phi|$ be the root system corresponding to H. Then* $\dim L = \dim H + |\Phi|$.

Proof. We can decompose L as a direct sum of root spaces,

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

By Proposition 10.9, each L_α is one dimensional. Thus

$$\dim L = \dim H + \dim L_{\alpha_1} + \dim L_{\alpha_2} + \ldots = \dim H + |\Phi|$$

$\square$

Proposition 15.6 (Exercise 10.6)**.** *Let* $L = \mathrm{sl}(3, \mathbb{C})$*, and let* $H = \mathrm{span}\{e_{11} - e_{22}, e_{22} - e_{33}\}$ *be the Cartan subalgebra of diagonal matrices. Then the set of roots for* H *is*

$$\Phi = \{\alpha, -\alpha, \beta, -\beta, \alpha + \beta, -\alpha - \beta\}$$

where $\alpha = \epsilon_1 - \epsilon_2, \beta = \epsilon_2 - \epsilon_3$*.*

Proof. (For the definition of ϵ_i see page 92.) As shown on page 92, if $i \neq j$ and

$$L_{ij} = \{x \in \mathrm{sl}(3, \mathbb{C}) : \mathrm{ad}\, h(x) = (\epsilon_i - \epsilon_j)(h)x, \forall h \in H\}$$

then $Lij = \mathrm{span}\{e_{ij}\}$ and L_{ij} is the root space for $\epsilon_i - \epsilon_j$. Thus

$$\begin{array}{lll} L_{12} = L_\alpha & L_{23} = L_\beta & L_{13} = L_{\alpha+\beta} \\ L_{21} = L_{-\alpha} & L_{32} = L_{-\beta} & L_{31} = L_{-\alpha-\beta} \end{array}$$

And thus

$$\mathrm{sl}(3, \mathbb{C}) = H \oplus \bigoplus_{i \neq j} L_{ij} = H \oplus \bigoplus_{\gamma \in \Phi} L_\gamma$$

$\square$

16 Chapter 11 Exercises

Proposition 16.1 (Exercise 11.1)**.** *In $\mathbb{R}^n$ with the usual dot product, let e_i be the vector with a 1 in the ith position and zeroes elsewhere. Define*

$$R = \{e_i - e_j : 1 \leq i \neq j \leq n\}$$
$$E = \operatorname{span} R$$

Then R is a root system for E.

Proof. (R1) Clearly R is finite, by definition of E, R spans E, and $0 \notin R$.
(R2) It is obvious from the definition of R that it doesn't contain any multiples of x other than $\pm x$ for $x \in R$.
(R4) Let $x = e_i - e_j, y = e_k - e_m$, so $x, y \in R$.

$$\langle x, y \rangle = \frac{2(x,y)}{(y,y)} = \frac{2(x,y)}{2} = (x,y) = (e_i - e_j) \cdot (e_k - e_m) = \delta_{ik} - \delta_{jk} - \delta_{im} + \delta_{jm} \in \mathbb{Z}$$

(R4) This one is much harder than the others. We need to show that for $x \in R$, s_x is a permutation of R. Since s_x is a reflection through a hyperplane, it is a bijection from $\mathbb{R}^n$ to $\mathbb{R}^n$. Thus all we need to show is that for $x, y \in R$, we have $s_x(y) \in R$. Let $x = e_i - e_j$ and $y = e_k - e_m$ ($i \neq j$ and $k \neq m$). Then

$$s_x(y) = x - \langle x, y \rangle = x - \frac{2x \cdot y}{y \cdot y} y = x - (x \cdot y)y$$

Now we have a bunch of cases.

$$\begin{aligned} x \cdot y &= \delta_{ik} - \delta_{im} + \delta_{jm} - \delta_{jk} \\ &= \begin{cases} 2 & j = m, i = k \\ 1 & i = k, j \neq m, j \neq k \text{ OR } i \neq k, i \neq m, j = m \\ 0 & i \neq k, i \neq m, j \neq k, j \neq m \\ -1 & i \neq k, i \neq m, j = k \text{ OR } i = m, j \neq m, j \neq k \\ 2 & i = m, j = k \end{cases} \end{aligned}$$

(Note that in the case where $x \cdot y = 0$, there might seem to be more possibilities, but those possibilities are ruled out since $i \neq j$ and $k \neq m$.) Thus in these same cases,

$$s_x(y) = \begin{cases} x - 2u & j = m, i = k \\ x - y & i = k, j \neq m, j \neq k \text{ OR } i \neq k, i \neq m, j = m \\ x & i \neq k, i \neq m, j \neq k, j \neq m \\ x + y & i \neq k, i \neq m, j = k \text{ OR } i = m, j \neq m, j \neq k \\ x + 2y & i = m, j = k \end{cases} = \begin{cases} (e_i - e_j) - 2(e_i - e_ij) = e_j - e_i \\ (e_i - e_j) - (e_i - e_m) = e_m - e_i \\ (e_i - e_j) - (e_k - e_j) = e_i - e_k \\ (e_i - e_j) \\ (e_i - e_j) + (e - j - e_m) = e_i - e_m \\ (e_i - e_j) + (e_k - e_i) = e_k - e_j \\ (e_i - e_j) + 2(e_j - e_i) = e_j - e_i \end{cases}$$

In every case, $s_x(y) \in R$, so (R4) $\square$

Proposition 16.2 (Exercise 11.2)**.** *Let R be a root system and let $\alpha, \beta \in R$ such that $(\alpha, \beta) \neq 0$. Then $(\alpha, s_\alpha(\beta)) \neq 0$.*

Proof.

$$\begin{aligned}(\alpha, s_\alpha(\beta)) &= \left(\alpha, \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha\right)\\ &= (\alpha,\beta) - \left(\alpha, \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha\right)\\ &= (\alpha,\beta) - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}(\alpha,\alpha)\\ &= (\alpha,\beta) - 2(\alpha,\beta)\\ &= -(\alpha,\beta)\end{aligned}$$

If $(\alpha, \beta) \neq 0$, then certainly $-(\alpha, \beta) \neq 0$. □

Proposition 16.3 (Exercise 11.2)**.** *Let R be a root system in inner product space E. Let R_i be the equivalence classes defined in Lemma 11.8. Then R_i satisfies (R3).*

Proof. We need to show that for $\alpha, \beta \in R_i$, we have $s_\alpha(\beta) \in R_i$. Let $\alpha, \beta \in R_i$. Then there exist $\gamma_1, \gamma_2, \dots \gamma_n$ such that $\alpha = \gamma_1, \beta = \gamma_2$, and $(\gamma_k, \gamma_{k+1}) \neq 0$ for $k = 1, 2, \dots (n-1)$. Since s_α preserves the inner product,

$$0 \neq (\gamma_k, \gamma_{k+1}) = (s_\alpha(\gamma_k), s_\alpha(\gamma_{k+1}))$$

Note that $s_\alpha(\gamma_1) = s_\alpha(\alpha) = -\alpha$. (Note that since $(-\alpha, s_\alpha(\gamma_2)) = -(\alpha, s_\alpha(\gamma_2))$, we get $(\alpha, s_\alpha(\gamma_2) \neq 0$.) Thus we have $\alpha, s_\alpha(\gamma_2), s_\alpha(\gamma_3), \dots s_\alpha(\gamma_n) = s_\alpha(\beta)$ such that

$$\begin{aligned}(\alpha, s_\alpha(\gamma_2)) &\neq 0\\ (s_\alpha(\gamma_2), s_\alpha(\gamma_3) &\neq 0\\ &\vdots\\ (s_\alpha(\gamma_{n-1}), s_\alpha(\beta)) &\neq 0\end{aligned}$$

Thus $s_\alpha(\beta) \sim \alpha$, so $s_\alpha(\beta) \in R_i$. □

Proposition 16.4 (Exercise 11.4)**.** *Let $R = \{\pm(e_i - e_j) : 1 \leq i < j \leq l+1\}$ and let $E = \operatorname{span} R$. ($E$ is a subspace of the inner product space $\mathbb{R}^{l+1}$}.) Let $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq l$. Let $B = \{\alpha_1, \dots \alpha_l\}$. Then B is a base for R.*

Proof. First we need to show that B is a basis for E. B spans E since the $-(e_i - e_j)$ contribute nothing to the span of R. We need to show that B is linearly independent. Suppose that

$$\sum_{i=1}^{l} c_i \alpha_i = 0$$

Then

$$\begin{aligned}
0 &= \sum_{i=1}^{l} c_i(e_i - e_{i+1}) \\
&= \sum_{i=1}^{l} c_i e_i - \sum_{i=1}^{l} c_i e_{i+1} \\
&= \sum_{i=1}^{l} c_i e_i - \sum_{i=2}^{l+1} c_{i-1} e_i \\
&= c_1 e_1 + (c_1 - c_2)e_2 + \ldots + (c_{l-1} - c_l)e_l - c_l e_{l+1}
\end{aligned}$$

Then since $\{e_1, \ldots e_{l+1}\}$ is linearly independent, it follows that $c_1 = 0, c_1 = c_2, c_3 = c_2, \ldots$ so we have $c_1 = c_2 = \ldots = c_{l+1} = 0$. Thus B is linearly independent, so it is a basis for E.

Now we need to show that every $\beta \in R$ can be written as

$$\sum_{\alpha \in B} k_\alpha \alpha$$

where the nonzero k_α have the same sign. Let $\beta \in R$. Then $\beta = \pm(e_i - e_j)$, where $1 \leq i < j \leq l + 1$. Then

$$\begin{aligned}
\beta &= \pm(e_i + (-e_{i+1} + e_{i+1}) + (-e_{i+2} + e_{i+2}) + \ldots + (-e_{j-1} + e_{j-1} - e_j)) \\
&= \pm((e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \ldots + (e_{j-1} - e_j)) \\
&= \pm(\alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}) \\
&= \pm \sum_{\alpha \in B} k_\alpha \alpha
\end{aligned}$$

where

$$k = \begin{cases} 1 & \alpha \in \{\alpha_i, \ldots, \alpha_{j-1}\} \\ 0 & \text{otherwise} \end{cases}$$

The $\pm$ distributes to all the k_α, so all the nonzero k_α have the same sign. Thus B is a base for R. $\square$

Proposition 16.5 (Exercise 11.4)**.** *Let R, B be as in the above proposition. Then the positive roots of R are*

$$\{e_i - e_j : i < j\}$$

(Every other root is negative.)

Proof. As shown above, if $\beta = e_i - e_j$ where $i < j$, then $\beta = \sum_\alpha k_\alpha \alpha$ where $k_\alpha \in \{0, 1\}$. Thus β is a positive root. If $\beta = -(e_i - e_j)$ then $\beta = -\sum_\alpha k_\alpha \alpha$, so β is a negative root. $\square$

Proposition 16.6 (Exercise 11.5)**.** *Let R be a root system with base B. Fixe $\gamma \in R$. Then*

$$s_\gamma(B) = \{s_\gamma(\alpha) : \alpha \in B\}$$

is a base for R.

Proof. We know that $s_\gamma : E \to E$ is a linear bijection, and B is a basis for E, so $s_\gamma(B)$ is a basis for E. We also need to show that for $\beta \in R$, β can be written as

$$\beta = \sum_{\alpha \in s_\gamma(B)} c_\alpha \alpha$$

where the nonzero c_α all have the same sign. Since s_γ permutes R, there exists $\beta_0 \in R$ such that $s_\gamma(\beta_0) = \beta$. Since B is a base, we can write β_0 as

$$\beta_0 = \sum_{\alpha \in B} k_\alpha \alpha$$

where the nonzero k_α have the same sign. By linearity of s_γ,

$$\beta = s_\gamma(\beta_0) = \sum_{\alpha \in B} k_\alpha s_\gamma(\alpha) = \sum_{\alpha \in s_\gamma(\beta)} k_\alpha \alpha$$

Thus we have written β in the necessary form. $\square$

17 Appendix A - Linear Algebra

17.1 Quotient Spaces

Definition 17.1. *Let V be a vector space over F and let W be a subset of V. Then W is a **subspace** of V if W is also a vector space over F (with the same addition as in V).*

Proposition 17.2 (Condition for being a subspace)**.** *Let V be a vector space over F and let $W \subset V$ such that for $v, u \in W$ and $t \in F$, $v + u \in W$ and $tv \in W$. Then W is a subspace of V.*

Proof. Let V, W, F be as described. Since $0v \in W$, W is nonempty. By hypothesis, W is closed under addition. Let $v \in W$. Then $(-1)v \in W$, so $-v \in W$, so W is also closed under taking inverses, so W is a subgroup of V as an Abelian group, so W is an Abelian group. Thus W satisfies properties 1-5 in the definition. Properties 6-10 follow for W since they hold for all elements of V and F. Thus W is a vector space over F. □

Definition 17.3. *Let V be a vector space over F and let W be a subspace of V. A **coset** of W is a set of the form*

$$v + W = \{v + w : w \in W\}$$

Proposition 17.4. *Let V be a vector space over F and let W be a subspace of V. Let $v, v' \in V$. Then the cosets $v + W$ and $v' + W$ are equal if and only if $v - v' \in W$.*

Proof. Suppose $v - v' \in W$. Let $w_0 = v - v'$, rearranging we have $v = w_0 + v'$. Let $x \in v + W$. Then $x = v + w_1$ for $w_1 \in W$, so $x = w_0 + w_1 + v'$. Since W is closed under addition, $w_0 + w_1 \in W$ so $x = v' + (w_0 + w_1) \in v' + W$. Likewise, suppose $y \in v' + W$. Then $y = v' + w_2 = w_0 + v + w_1 = v + (w_0 + w_2) \in v + W$. Thus if $v - v' \in W$, then $v + W = v' + W$.

Now suppose $v + W = v' + W$. Since $0 \in W$, $v + 0 = v \in v + W$, and since $v + W = v' + W$, $v \in v' + W$. Then there exists w_0 such that $v = v' + w_0$. Then $w_0 = v - v'$, so $v - v' \in W$. □

Definition 17.5. *Let V be a vector space over F and let W be a subspace of V. The **quotient space** V/W is the set of all cosets of W, that is, the set*

$$V/W = \{v + W : v \in V\}$$

We then define addition in this space by

$$(v + W) + (v' + W) = (v + v') + W$$

where $v, v' \in V$. We define scalar multiplication from F by

$$\lambda(v + W) = \lambda v + W$$

where $\lambda \in F$.

Proposition 17.6. *Addition and scalar multiplication for quotient spaces are well-defined.*

Proof. First we show that scalar multiplication is well-defined. Let $v, v' \in V$ such that $v+W = v'+W$. We need to show that for $\lambda \in F$, $\lambda(v+W) = \lambda(v'+W)$. Since $v+W = v'+W$, by Proposition 17.4, $v - v' \in W$. Since W is closed under scalar multiplication, $\lambda(v - v') = \lambda v - \lambda v' \in W$. Then using the other direction of Proposition 17.4, $\lambda v + W = \lambda v' + W$, so we have shown what was needed to show, since $\lambda v + W = \lambda(v + W)$ by definition.

Now we will show that addition is well-defined. Let $v_1, v_2 \in V$. We wil show ow that the addition $(v_1 + W) + (v_2 + W)$ does not depend on coset representative. Let $v_1', v_2' \in V$ such that $v_1'+W = v_1+W$ and $v_2'+W = v_2+W$. Now we need to show that $(v_1+w)+(v_2+W) = (v_1'+W)+(v_2'+W)$. By Proposition 17.4, $v_1' = v_1+w_1$ and $v_2' = v_2+w_2$ for some $w_1, w_2 \in W$. Note that since W is closed under addition, $w_1 + w_2 \in W$, soo

$$\begin{aligned}(v_1' + W) + (v_2' + W) &= (v_1' + v_2') + W \\ &= ((v_1 + w_1) + (v_2 + w_2)) + W \\ &= (v_1 + v_2 + (w_1 + w_2)) + W \\ &= (v_1 + v_2) + W \\ &= (v_1 + W) + (v_2 + W)\end{aligned}$$

□

Proposition 17.7. *Let V be a vector space over F, and let W be a subspace. Then the quotient space V/W is a vector space over F.*

Proof. Let Let $v, u, w \in V$ and $a, b \in F$, so $v + W, u + W \in V/W$.
(Closure) By closure of addition in V, $v+u \in V$ so $(v+W)+(u+W) = (v+u)+W \in V/W$.
(Associativity of addition) By associativity of addition in V,

$$\begin{aligned}(v + W) + \big((u + W) + (w + W)\big) &= (v + W) + \big((u + w) + W\big) \\ &= \big(v + (u + w)\big) + W \\ &= \big((v + u) + w\big) + W \\ &= \big((v + W) + (u + W)\big) + (w + W)\end{aligned}$$

(Commutativity of addition) By commutativity of addition in V,

$$(v + W) + (u + W) = (v + u) + W = (u + v) + W = (u + W) + (v + W)$$

(Identity for Addition) $0 + W = W$ is the identity for addition because

$$(0 + W) + (v + W) = (0 + v) + W = v + W$$

(Inverses for Addition) $(-v + W)$ is an additive inverse for $(v + W)$ because

$$(-v + W) + (v + W) = (-v + v) + W = 0 + W$$

(Closure of scalar multiplication) For $v \in V, a \in F$, $av \in V$ by closure of scalar multiplication in V, so $a(v + W) = av + W \in V/W$.
(Associativity of scalar multiplication) $a(b(v+W))+a(bv+W) = (ab)v+W = (ab)(v+W)$.

(Identity for scalar multiplication) 1 (the multiplicative identity for F) is the identity for V/W also, since $1(v+W) = 1v + W = v + W$.
(Distributivity over vector sums)

$$\begin{aligned} a\big((v+W)+(u+W)\big) &= a\big((v+u)+W\big) \\ &= a(v+u)+W \\ &= (av+au)+W \\ &= (av+W)+(au+W) \end{aligned}$$

(Distributivity over scalar sums)

$$\begin{aligned} (a+b)(v+W) &= (a+b)v+W \\ &= (av+bv)+W \\ &= (av+W)+(bv+W) \end{aligned}$$

□

Proposition 17.8. *Let V be a vector space with subspace W. If $v_1, v_2, \dots v_k$ are vectors in V such that the cosets $v_1+W, v_2+W, \dots v_3+W$ form a basis for the quotient space V/W, then $v_1 \dots v_k$ together with any basis for W forms a basis for V.*

Proof. Asserted on page 190 of Erdmann and Wildon. □

17.2 Linear Maps

Lemma 17.9. *Let V, W be vector spaces over a field F and let $\phi : V \to W$ be an onto linear map. Let β be a basis for V. Then $\phi(\beta)$ is a spanning set for W.*

Proof. Let $\beta = \{v_1, \dots v_n\}$. We need to show that any $w \in W$ can be written as a linear combination of $\{\phi(v_1), \dots \phi(v_n)\}$. Let $w \in W$. Since ϕ is onto, there exists $v \in V$ such that $\phi(v) = w$. We can write v as a unique linear combination of the basis vectors, $v = \sum a^i v_i$ where $a^i \in F$. Then

$$w = \phi(w) = \phi\left(\sum a^i v_i\right) = a^i \sum \phi(v_i)$$

Thus $\phi(\beta)$ is a spanning set for W. □

Lemma 17.10. *Let V, W be vector spaces over a field F and let $\phi : V \to W$ be a one-to-one linear map. Let β be a basis for V. Then $\phi(\beta)$ is linearly independent.*

Proof. Let $\beta = \{v_1 \dots v_n\}$. Suppose that

$$\sum a^i \phi(v_i) = 0$$

for $a^i \in F$. Then by linearity of ϕ,

$$\phi\left(\sum a^i v_i\right) = 0$$

Since ϕ is one-to-one, the kernel is only zero, so $\sum a^i v_i = 0$. Since β is a basis, this implies that $a^i = 0$ for all i. Thus $\phi(\beta)$ is linearly independent. □

Corollary 17.11. *Let V, W be vectors spaces over a field F and let $\phi : V \to W$ be a linear bijection. Let β be a basis for V. Then $\phi(\beta)$ is a basis for W.*

Proof. By the previous lemmas, since ϕ is one-to-one and onto, $\phi(\beta)$ is a linearly independent spanning set, so it is a basis. $\square$

Proposition 17.12. *Let $\phi : V \to W$ be a linear map, where V and W are vector spaces over F. Then $\ker \phi$ is a subspace of V.*

Proof. Let $v_1, v_2 \in \ker \phi$. Then since ϕ is a linear map, $\phi(v_1+v_2) = \phi(v_1)+\phi(v_2) = 0+0 = 0$, so $v_1 + v_2 \in \ker \phi$. Let $a \in F$. Then $\phi(av_1) = a\phi(v_1) = (a)0 = 0$ so $av_1 \in \ker \phi$. Then since $\ker \phi$ is closed under vector addition and s $\square$

Proposition 17.13 (Rank-Nullity Theorem)**.** *Let V, W be finite-dimensional vector spaces over F and let $\phi : V \to W$ be a linear map. Then*

$$\dim V = \dim \operatorname{im} \phi + \dim \ker \phi \tag{17.1}$$

Proof. (Proof by Lawrence Valby) Let $\phi : V \to W$ be a linear map and let $n = \dim V$. By Proposition 17.12, $\ker \phi$ is a subspace of V. Since $\ker \phi$ is a vector space, it has a basis $B = \{b_1, b_2, \dots b_k\}$ where $k = \dim \ker \phi$, and where $k \leq n$. If $k = n$, then $\ker \phi = V$ and $\dim \operatorname{im} \phi = 0$ so the result is true, so assume that $k \neq n$. Since B is a linearly independent subset of V, we can extend it to a basis for V, finding $C = \{c_1, c_2 \dots c_{n-k}\}$ where $B \cup C$ is a basis for V. We claim that $\phi(C) = \{\phi(c_1), \phi(c_2), \dots \phi(c_{n-k})\}$ is a basis for $\operatorname{im} \phi$. If it is, then $\dim \operatorname{im} \phi = n - k$ and then $\dim \operatorname{im} \phi + \dim \ker \phi = (n - k) + k = n = \dim V$.

First we show that $\phi(C)$ spans $\operatorname{im} \phi$. Let $w \in \operatorname{im} \phi \subseteq W$. Then there exists $v \in V$ such that $\phi(v) = w$. Since $B \cup V$ spans V, there exist scalars $r_1, r_2, \dots r_k \in F$ and $s_1, s_2 \dots s_{n-k} \in F$ such that $v = \sum_{i=1}^{k} r_i b_i + \sum_{i=1}^{n-k} s_i c_i$. Then by the linear property of ϕ,

$$\phi(v) = \phi\left(\sum_{i=1}^{k} r_i b_i + \sum_{i=1}^{n-k} s_i c_i\right) = \sum_{i=1}^{k} r_i \phi(b_i) + \sum_{i=1}^{n-k} s_i \phi(c_i) = \sum_{i=1}^{n-k} s_i \phi(c_i)$$

since $b_i \in \ker \phi$ for all i. Thus, $\phi(v) = w$ can be written as a linear combination of elements in $\phi(C)$, so $\phi(C)$ spans $\operatorname{im} \phi$.

Now we show that $\phi(C)$ is linearly independent. Let $s_1, s_2 \dots s_{n-k} \in F$ be scalars such that $\sum_{i=1}^{n-k} s_i \phi(c_i) = 0$. Then if we show that $s_1 = s_2 = \dots = s_{n-k} = 0$ we have show that $\phi(C)$ is linearly independent. By the linear property of ϕ, $\phi(\sum_{i=1}^{n-k} s_i c_i) = 0$ so $\sum_{i=1}^{n-k} s_i c_i \in \ker \phi$. Since B spans $\ker \phi$, there exist scalars $r_1, r_2 \dots r_k \in F$ such that

$$\sum_{i=1}^{k} r_i b_i = \sum_{i=1}^{n-k} s_i c_i$$

We can rearrange this to give

$$\sum_{i=1}^{k} (-r_i) b_i + \sum_{i=1}^{n-k} s_i c_i = 0$$

Since $B \cup C$ is a basis, it is linearly independent, so all the scalars r_i, s_i are equal to zero. Thus $\phi(C)$ is linearly independent.

We have show that $\phi(C)$ spans $\operatorname{im}\phi$ and is linearly independent, so it is a basis for $\operatorname{im}\phi$. Thus $\dim \operatorname{im}\phi = n-k$, so $\dim \operatorname{im}\phi + \dim\ker\phi = (n-k)+k = n = \dim V$. $\square$

Proposition 17.14. *Let V be a vector space over a field F and let U, W be subspaces of V. Then $U+W$ is a subspace of V.*

Proof. Using Propositions 17.2, we must show that $U+W$ is closed under vector addition and scalar multiplication, then it will be shown that it is a subspace.

Let $v_1, v_2 \in U+W$. Then by definition of $U+W$ there exist u_1, u_2, w_1, w_2 such that $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$. Then $v_1 + v_2 = (u_1+w_1)+(u_2+w_2) = (u_1+u_2)+(w_1+w_2)$. Since U, W are subspaces, $u_1 + u_2 \in U$ and $w_1 + w_2 \in W$, thus $v_1 + v_2 \in U+W$.

Let $v_1 \in U+W$ and $t \in F$. Then $v_1 = u_1 + w_1$ for some $u_1 \in U, w_1 \in W$. Then $tv_1 = tu_1 + tw_1$. Since U, W are subspaces, $tu_1 \in U$ and $tw_1 \in W$, thus $tv_1 \in U+W$. $\square$

17.3 Matrices and Diagonalisation

Definition 17.15. *Let V be an n-dimensional vector space over F with $\beta = \{v_1, v_2, \ldots v_n\}$ a basis for V. For $v \in V$, we can write v uniquely as a linear combination of basis elements, $v = \sum_{i=1}^n a^i v_i$ where $a^i \in F$. We define a map $[\ ]_\beta : V \to F^n$ by $[v]_\beta = (a^1, a^2, \ldots a^n)$.*

Definition 17.16. *Let V be an n-dimensional vector space over F with basis β. Let $x : V \to V$ be a linear map. Then the matrix of x is $[x]_\beta$, the unique matrix in $\mathrm{gl}(n,F)$ satisfying*

$$[x]_\beta [v]_\beta = [x(v)]_\beta$$

for all $v \in V$. We usually suppress the subset β since the choice of basis is clear from context, writing

$$[x][v] = [x(v)]$$

Lemma 17.17. *Let V be an n-dimensional vector space over F with basis $\beta = \{v_1, v_2, \ldots v_n\}$. Let $x : V \to V$ be a linear map, and $[x] = (a_{ij})$ be the matrix of x with respect to β. Then*

$$x(v_j) = \sum_{i=1}^n a_{ij} v_i$$

for $v_j \in \beta$.

Proof. Let $v_j \in \beta$. Then $[v_j] = (\delta_{ij})_{i=1}^n = (0, 0, \ldots 1, \ldots 0, 0)$ where the 1 is in the jth position. By the definition of $[x]$,

$$[x(v_j)] = [x][v_j] = (a_{ij})(\delta_{ij}) = (a_{1j}, a_{2j}, \ldots a_{nj})$$

So we have $[x(v_j)] = (a_{1j}, a_{2j}, \ldots a_{nj})$. Thus by the definition of the map $[\ ]_\beta$,

$$x(v_j) = \sum_{i=1}^n a_{ij} v_i$$

$\square$

Lemma 17.18. *Let V be an n-dimensional vector space over F with basis $\beta = \{v_1, \dots v_n\}$. Let $v, w \in V$ and let $\lambda \in F$. Then*

$$[v]_\beta + [w]_\beta = [v + w]_\beta$$
$$\lambda[v]_\beta = [\lambda v]_\beta$$

Proof. Let $v = \sum_i a^i v_i$ and let $w = \sum_i b^i v_i$. Then by definition,

$$[v]_\beta = (a^1, \dots a^n)$$
$$[w]_\beta = (b^1, \dots b^n)$$

Then

$$\begin{aligned} [v]_\beta + [w]_\beta &= (a^1, \dots a^n) + (b^1, \dots b^n) = (a^1 + b^1, \dots a^n + b^n) \\ v + w &= \sum_i a^i v_i + \sum_i b^i v_i = \sum_i (a^i + b^i) v_i \\ [v + w]_\beta &= (a^1 + b^1, \dots a^n + b^n) \end{aligned}$$

Thus $[v]_\beta + [w]_\beta = [v + w]_\beta$. Also,

$$\lambda[v]_\beta = \lambda(a^1, \dots a^n) = (\lambda a^1, \dots \lambda a^n) = [\lambda v]_\beta$$

□

Lemma 17.19. *Let V be an n-dimensional vector space over F with basis β and let $x, y : V \to V$ be linear maps and let $\lambda \in F$. Then*

$$[x + y]_\beta = [x]_\beta + [y]_\beta$$
$$[\lambda x]_\beta = \lambda[x]_\beta$$

Proof. Let $v \in V$. Then

$$\begin{aligned} [x + y]_\beta &= [(x + y)(v)]_\beta = [x(v) + y(v)]_\beta = [x(v)]_\beta + [y(v)]_\beta \\ &= [x]_\beta[v]_\beta + [y]_\beta[v]_\beta = ([x]_\beta + [y]_\beta)[v]_\beta \end{aligned}$$

Thus $[x + y]_\beta = [x]_\beta + [y]_\beta$. To show the scalar mutliplication property, observe that

$$[\lambda x][v] = [(\lambda x)(v)] = \lambda[x(v)] = \lambda[x][v]$$

□

Proposition 17.20 (Exercise 16.1i)**.** *Let V be a n-dimensional vector space with basis β and let $x, y : V \to V$ be linear maps. Then*

$$[y \circ x]_\beta = [y]_\beta [x]_\beta$$

Proof. We omit the subscript βs for clarity. By definition of $[x]$ and $[y]$ we have

$$[x][v] = [x(v)]$$
$$[y][v] = [y(v)]$$

for all $v \in V$. Then since $x(v) \in V$,

$$[y]\big([x][v]\big) = [y][x(v)] = [y(x(v)] = [y \circ x(v)]$$

By definition of $[y \circ x]$,

$$[y \circ x][v] = [y \circ x(v)]$$

Thus $[y \circ x] = [y][x]$. □

Lemma 17.21. *Let A, B be similar matrices. Then the set of eigenvalues for A is equal to the set of eigenvalues for B.*

Proof. Since A, B are similar, there exists an invertible matrix P such that $A = PBP^{-1}$. Let λ be an eigenvalue of A. Then $Av = \lambda v$ for some vector v. Then $PBP^{-1}v = \lambda v$, so $B(P^{-1}v) = P^{1}\lambda v = \lambda(P^{-1}v)$. Thus λ is an eigenvalue of B, with corresponding eigenvector $P^{-1}v$. □

Proposition 17.22 (Exericse 16.2)**.** *Let $x \in \mathrm{gl}(V)$, and let $f(t)$ be a polynomial with $f(x) = 0$. Then $m_x(t)$ (the minimal polynomial of x) divides $f(t)$.*

Proof. The Euclidean division algorithm for polynomials says that there exist polynomials $q(t), r(t)$ with $\deg r(t) < \deg m_x(t)$ such that

$$f(t) = q(t)m_x(t) + r(t)$$

We can rearrange this to get $r(t) = f(t) - q(t)m_x(t)$. From this, we also get $r(x) = f(x) - q(x)m_x(x)$. By the definition of m_x, $m_x(x) = 0$, and by hypothesis $f(x) = 0$, so $r(x) = 0$. Since $m_x(t)$ is the lowest degree polynomial that kills x, and $r(x) = 0$ and $\deg r(t) < \deg m_x(t)$, it must be that $r(t) = 0$. Thus $f(t) = q(t)m_x(t)$, so $m_x(t)$ divides $f(t)$. □

Proposition 17.23 (Lemma 16.7)**.** *Let V be a vector space and let $x_1, x_2, \ldots x_k : V \to V$ be diagonalizable linear maps. Then there exists a basis β of V that simultaneously diagnalizes each x_i if and only if for each i, j, $x_i \circ x_j = x_j \circ x_i$.*

Proof. From now on when writing a composition of maps, we omit the $\circ$ and simply write $x_i x_j$. Suppose that there is a basis β that simultaneously diagonalizes each x_i. Then

$$[x_i x_j]_\beta = [x_i]_\beta [x_j]_\beta = [x_j]_\beta [x_i]_\beta = [x_j x_i]_\beta$$

The matrix representations $[x_i]_\beta, [x_j]_\beta$ commute because they are diagonal matrices, and since the matrix representation of $x_i x_j$ and $x_j x_i$ are equal, they are equal as linear maps. This completes the easier direction of the proof.

Now suppose that $x_1, x_2, \ldots x_k$ all pairwise commute. We will show that they are all simultaneously diagonalizable. We proceed by induction on k. The base case is $k = 2$. Suppose x_1, x_2 are diagonalizable, commuting linear maps. Since x_1 is diagonalizable, we can write V as a direct sum of eigenspaces for x_1.

$$V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \ldots \oplus V_{\lambda_r}$$

If $v \in V_{\lambda_i}$, then $x_1(v) = \lambda_i v$. We also claim that $w = x_2(v) \in V_{\lambda_i}$.

$$x_1(w) = x_1(x_2(v)) = x_2(x_1(v)) = x_2(\lambda_i v) = \lambda_i x_2(v) = \lambda_i w$$

so we have $x_1(w) = \lambda_i w$, so w is an eigenvector of x_1 with eigenvalue λ_i, so $w = x_2(v) \in V_{\lambda_i}$ by definition of V_{λ_i}. Thus V_{λ_i} is an x_2 invariant space. By Corollary 16.5(a), x_2 restricted to V_{λ_i} is diagonalizable. Thus there is a basis γ_i of V_{λ_i} consisting of eigenvectors for x_2. If $v \in \gamma_i$, then v is a linear combination of eigenvectors of x_i with eigenvalue λ_i, so v is an eigenvector of x_1 with eigenvalue λ_i. Hence

$$\gamma = \gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_r$$

is a basis of eigenvectors for both x_1 and x_2. Thus x_1, x_2 are simultaneously diagonalized by the basis γ of V. The completes the base case.

For the inductive step. suppose that if $x_1, x_2, \ldots x_k$ commute then they are simultaneously diagonalizable. Suppose $x_1, x_2, \ldots x_{k+1}$ commute. Then there is a basis β that simultaneously diagonalizes $x_1, \ldots x_k$. Consider the composition $x_1 x_2 \ldots x_k$. It has matrix representation

$$\left[\prod_{i=1}^{k} x_i\right]_\beta = \prod_{i=1}^{k} [x_i]\beta$$

But each $[x_i]_\beta$ is diagonal, so the matrix of the product is also diagonal. Since x_{k+1} commutes with each x_i, it commutes with the product, so by the base case, there is a basis Ω that simultaneously diagonalizes this product and x_{k+1}. □

17.4 Interlude: The Diagonal Fallacy

Proposition 17.24 (Exercise 16.3)**.** *Let V be a 2-dimensional vector space with basis $\beta = \{v_1, v_2\}$. Let $x : V \to V$ be a linear map with matrix*

$$[x]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

If U is a subspace of V such that $x(U) \subseteq U$, then $U = \{0\}, U = V$, or $U = \operatorname{span}\{v_1\}$.

Proof. One can check that $x(\{0\} = \{0\}$ and $x(V) = \operatorname{span}\{v_1\} \subseteq V$ and $x(\operatorname{span}\{v_1\}) = \{0\} \subseteq \operatorname{span}\{v_1\}$. This deals with all zero- and two-dimensional subspaces of V. We must show that no other 1-dimensional subspace of V is x-invariant. Let $w = a^1 v_1 + a^2 v_2$. Then any 1-dimensional subspace of V will be of the form $\operatorname{span}\{w\}$. Then

$$[x(w)]_\beta = [x]_\beta [w]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} = \begin{pmatrix} a^2 \\ 0 \end{pmatrix}$$

Thus $x(\text{span}\{w\}) = \text{span}\{v_1\}$. So if $\text{span}\{w\}$ is x-invariant, it must be that $w \in \text{span}\{v_1\}$, so it must be that $\text{span}\{w\} = \text{span}\{v_1\}$. Thus the only 1-dimensional subspace of V that is x-invariant is $\text{span}\{v_1\}$. □

Proposition 17.25 (Exercise 16.6i)**.** *Let V be a vector space over a field F. Let $\text{Hom}(V, V) = \{x : V \to V\}$ be the set of linear maps from V to itself, and for $x, y \in \text{Hom}(V, V)$ and $\lambda \in F$ define*

$$(x + y)v = x(v) + y(v)$$
$$(\lambda x)v = \lambda(x(v))$$

Then $\text{Hom}(V, V)$ is a vector space over F under these operations. It has dimension $(\dim V)^2$.

Proof. Closure of addition - $x + y$ is clearly linear. Inverse - $(x + (-x))v = x(v) - x(v) = 0$. Associativity - from associativity in V. Commutativity - from commutativity in V. Identity - zero map, $0(v) = 0$. To see that the dimension of $\text{Hom}(V, V)$ is $(\dim V)^2$, note that there is a simple bijection between $\text{Hom}(V, V)$ and the set of $\dim V \times \dim V$ matrices with entries in F. □

Proposition 17.26 (Exercise 16.10)**.** *Let V be a finite-dimensional vector space over F and let $\text{Hom}(V, V)$ be the vector space of linear transformations from V to itself. Define*

$$\beta : \text{Hom}(V, V) \times \text{Hom}(V, V) \to F$$

by $\beta(x, y) = \text{tr}(xy)$. Then β is a symmetric, non-degenerate, bilinear form.

Proof. β is symmetric because $\text{tr}(xy) = \text{tr}(yx)$. β is linear because the trace function is linear:

$$\beta(ax + y, z) = \text{tr}((ax + y)z)) = a\,\text{tr}(xz) + \text{tr}(yz) = a\beta(x, y) + \beta(y, z)$$

To show that β is non-degenerate, we need to show that $\text{Hom}(V, V)^\perp = \{0\}$.

$$\text{Hom}(V, V)^\perp = \{x \in \text{Hom}(V, V) : \text{tr}(xy) = 0 \forall y \in \text{Hom}(V, V)\} = \{0\}$$

(According to the Wikipedia page on trace, "This follows from the fact that $\text{tr}(A^*A) = 0$ if and only if $A = 0$.") Thus β is non-degenerate. □

18 Connection to Lie Groups

Definition 18.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable and let $v \in \mathbb{R}^n$ with $v = (v^1, v^2, \dots v^n)$. Then $df : \mathbb{R}^n \to T\mathbb{R}^n$ is defined by*

$$df(v) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} v^i$$

Definition 18.2. *Let $U \subset \mathbb{R}^m$ and let $F : U \to \mathbb{R}^n$ be differentiable. Let $\alpha : I \to U$ be a smooth curve with $\alpha(0) = x$ and $\alpha'(0) = v$. Then $dF_x : U \to T_{F(x)}\mathbb{R}^n$ is defined by*

$$dF_x(v) = (F \circ \alpha)'(0)$$

Proposition 18.3 (Exercise 1.3.1a)**.** *Let $U \subset \mathbb{R}^m$ and let $F : U \to \mathbb{R}^n$ be a differentiable map. Let $x \in U$ and $v \in T_x\mathbb{R}^m$ be a tangent vector. Let $I \subset \mathbb{R}$ be an interval, and let $\alpha, \beta : I \to U$ be smooth curves with*

$$\alpha(0) = \beta(0) = x \qquad \alpha'(0) = \beta'(0) = v$$

Then $(F \circ \alpha)'(0) = (F \circ \beta)'(0)$.

Proof. We know that $F(x) = (f^1(x), f^2(x), \dots f^n(x))$ for differentiable functions $f^i : U \to \mathbb{R}$. Since $f^i \circ \alpha, f^i \circ \beta : I \to \mathbb{R}$ are differentiable function from $\mathbb{R}$ to $\mathbb{R}$, we can use the one-dimensional chain rule to get

$$\begin{aligned}(f^i \circ \alpha)'(t) &= \alpha'(t)((f^i)' \circ \alpha)(t)\\(f^i \circ \beta)'(t) &= \beta'(t)((f^i)' \circ \beta)(t)\end{aligned}$$

So $v(f^i)'(x) = (f^i \circ \alpha)'(0) = (f^i \circ \beta)'(0)$. Now we compute

$$\begin{aligned}(F \circ \alpha)(t) &= \big((f^1 \circ \alpha)(t), \dots (f^n \circ \alpha)(t)\big)\\(F \circ \alpha)'(t) &= \big((f^1 \circ \alpha)'(t), \dots (f^n \circ \alpha)'(t)\big)\\(F \circ \alpha)'(0) &= \big((f^1 \circ \alpha)'(0), \dots (f^n \circ \alpha)'(0)\big)\\&= \big((\alpha'(0)((f^1)' \circ \alpha)(0), \dots (\alpha'(0)((f^n)' \circ \alpha)(0)\big)\\&= \big((\beta'(0)((f^1)' \circ \beta)(0), \dots (\beta'(0)((f^n)' \circ \beta)(0)\big)\\&= \big((f^1 \circ \beta)'(0), \dots (f^n \circ \beta)'(0)\big)\\&= (F \circ \beta)'(0)\end{aligned}$$

□

Lemma 18.4 (for Exercise 1.3.1b)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then $df(v + w) = df(v) + df(w)$.*

Proof. Let $v = (v^1, v^2, \dots v^n), w = (w^1, w^2, \dots w^n)$.

$$\begin{aligned}df(v + w) &= df((v^1 + w^1, \dots v^n + w^n))\\&= \frac{\partial f}{\partial x^i}(v^i + w^i)\\&= \frac{\partial f}{\partial x^i} v^i + \frac{\partial f}{\partial x^i} w^i\\&= df(v) + df(w)\end{aligned}$$

$\square$

Proposition 18.5 (Exercise 1.3.1b). *Let $U \subset \mathbb{R}^m$ and let $F : U \to \mathbb{R}^n$ be differentiable. Let $x \in U$ and $v, w \in T_x\mathbb{R}^m$. Then*

$$dF_x(v + w) = dF_x(v) + dF_x(w)$$

Proof. Let $F(x) = (f^1(x), f^2(x), \dots f^n(x))$ for some differentiable functions $f^i : U \to \mathbb{R}$. Let $\alpha, \beta : I \to U$ be smooth curves such that $\alpha(0) = \beta(0) = x$ and $\alpha'(0) = v$ and $\beta'(0) = w$. Then

$$\begin{aligned}
dF_x(v) &= (F \circ \alpha)'(0) \\
&= \left(\frac{\partial}{\partial t} f^1 \circ \alpha(t), \frac{\partial}{\partial t} f^2 \circ \alpha(t), \dots \frac{\partial}{\partial t} f^n \circ \alpha(t) \right) \\
&= \big(df^1(\alpha'(0)), df^2(\alpha'(0)), \dots df^n(\alpha'(0)) \big)
\end{aligned}$$

And likewise for β,

$$dF_x(w) = (F \circ \beta)'(0) = \big(df^1(\beta'(0)), df^2(\beta'(0)), \dots df^n(\beta'(0)) \big)$$

Then we add them together and get

$$\begin{aligned}
dF_x(v) + dF_x(w) &= (F \circ \alpha)'(0) + (F \circ \beta)'(0) \\
&= (df^1(\alpha'(0)), df^2(\alpha'(0)), \dots df^n(\alpha'(0))) \\
&\quad + (df^1(\beta'(0)), df^2(\beta'(0)), \dots df^n(\beta'(0))) \\
&= (df^1(\alpha'(0)) + df^1(\beta'(0)), \dots df^n(\alpha'(0)) + df^n(\beta'(0))) \\
&= (df^1(\alpha'(0) + \beta'(0)), \dots df^n(\alpha'(0) + \beta'(0))) \\
&= (df^1(v + w), \dots df^n(v + w)) \\
&= dF_x(v + w)
\end{aligned}$$

$\square$

Proposition 18.6 (Exercise 1.3.1b). *Let $U \subset \mathbb{R}^m$ and let $F : U \to \mathbb{R}^n$ be differentiable. Let $x \in U$ and $v \in T_x\mathbb{R}^m$, $c \in \mathbb{R}$. Then*

$$dF_x(cv) = c\, dF_x(v)$$

Proof. Let $f^1, f^2, \dots f^n$ be the component functions of F, that is, $F(x) = (f^1(x), f^2(x), \dots f^n(x))$. Let $\alpha : I \to \mathbb{R}^m$ be a smooth curve such that $\alpha(0) = x$ and $\alpha'(0) = v$. Defined $\beta : I \to \mathbb{R}^m$ by $\beta(t) = \alpha(ct)$. Then $\beta(0) = x$ and $\beta'(0) = cv$, so $dF_x(cv) = (F \circ \beta)'(0)$. First we do some preliminary calculations.

$$\begin{aligned}
(f^i \circ \beta)(t) &= f^i(\beta(t)) = f^i(\alpha(ct)) = (f^i \circ \alpha)(ct) \\
(f^i \circ \beta)'(t) &= (f^i \circ \alpha)'(ct) = c(f^i \circ \alpha)'(t) \\
(F \circ \beta)(0) &= ((f^1 \circ \beta)(0), \dots (f^n \circ \beta)(0))
\end{aligned}$$

Now we can use these to evaluate $dF_x(cv)$.

$$\begin{aligned} dF_x(cv) &= (F \circ \beta)'(0) \\ &= ((f^1 \circ \beta)'(0), \dots (f^n \circ \beta)'(0)) \\ &= (c(f^1 \circ \alpha)'(0), \dots c(f^n \circ \alpha)'(0)) \\ &= c(F \circ \alpha)'(0) \\ &= c\, dF_x(v) \end{aligned}$$

□

Let G be a Lie group. Then for $g \in G$ we can define maps $L_g : G \to G$ by $L_g(h) = gh$. For $h \in G$, the differential $(dL_g)_h$ of L_g at h is a linear map $T_hG \to T_{gh}G$.

Proposition 18.7 (Exercise 1.4.3a)**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $v \in \mathfrak{g}$ and let $\tilde{v}$ be the vector field defined by*

$$\tilde{v}(g) = (dL_g)_e(v)$$

Then for $g, h \in G$,

$$\tilde{v}(gh) = dL_g(\tilde{v}(h))$$

Proof. We know that $L_{gh} = L_g \circ L_h$. Then since $L_g, L_h : G \to G$ are differentiable, we can use the Chain Rule,

$$d(L_{gh})_e = d(L_g \circ L_h)_e = (dL_g)_h \circ (dL_h)_e$$

Now we can compute

$$\begin{aligned} \tilde{v}(gh) &= (dL_{gh})_e(v) \\ &= ((dL_g)_h \circ (dL_h)_e)(v) \\ &= (dL_g)_h((dL_h)_e(v)) \\ &= (dL_g)_h(\tilde{v}(h)) \\ &= dL_g(\tilde{v}(h)) \end{aligned}$$

□

Proposition 18.8 (Exercise 1.4.4)**.** *Let $V = a^i \frac{\partial}{\partial x^i}, W = b^j \frac{\partial}{\partial x^j}$ be vector fields on a Lie group G. Then $[V, W]$ is a vector field on G, and in particular,*

$$[V, W] = c^j \left(a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

Proof. Let $f : G \to \mathbb{R}$ be a smooth function. Then

$$\begin{aligned} [V, W](f) &= V(W(f)) - W(V(f)) \\ &= V\left(a^i \frac{\partial}{\partial x^i} \right) - W\left(b^j \frac{\partial}{\partial x^j} \right) \\ &= a^i \frac{\partial}{\partial x^i} \left(a^i \frac{\partial}{\partial x^i} \right) - b^j \frac{\partial}{\partial x^j} \left(b^j \frac{\partial}{\partial x^j} \right) \end{aligned}$$

Now we use the product rule.

$$= a^i \left(\frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^j} + b^j \frac{\partial^2 f}{\partial x^i \partial x^j} \right) - b^j \left(\frac{\partial a^i}{\partial x^j} \frac{\partial f}{\partial x^i} + a^i \frac{\partial^2 f}{\partial x^j \partial x^i} \right)$$

By Clairaut's Theorem,

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$$

so we get nice cancellation of the second term of each sum. Thus

$$\begin{aligned} [V, W](f) &= a^i \frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^j} - b^j \frac{\partial a^i}{\partial x^j} \frac{\partial f}{\partial x^i} \\ &= a^i \frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^j} - b^i \frac{\partial a^j}{\partial x^i} \frac{\partial f}{\partial x^j} \\ &= \left(a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} f \end{aligned}$$

Thus we have written $[V, W](f)$ as a linear combination of $\frac{\partial}{\partial x^j} f$, and we know that $\{\frac{\partial}{\partial x^j}\}$ is a basis for the tangent space, so

$$[V, W] = \left(a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

□

Proposition 18.9. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and identity e. Let V be a left-invariant vector field on G, that is, $V : G \to TG$ with $V(gh) = (dL_g)_h(V(h))$. (Note that $(dL_g)_h : T_hG \to T_{gh}G$.) Let $v = V(e)$. Then $V = \tilde{v}$, where $\tilde{v} : G \to TG$ is defined by $\tilde{v}(g) = (dL_g)_e(v)$.*

Proof. To show: For $g \in G$, $V(g) = \tilde{v}(g)$.

$$\begin{aligned} \tilde{v}(g) &= (dL_g)_e(v) = (dL_g)_e(V(e)) \\ V(g) = V(ge) &= (dL_g)_e(V(e)) \end{aligned}$$

□

What the previous proposition shows is that every left-invariant vector field on G is equal to $\tilde{v}$ for some $v \in \mathfrak{g}$.
In the next proposition, $M(n, \mathbb{R})$ refers to the set of all $n \times n$ matrices with real entries and $\mathrm{GL}(n, \mathbb{R})$ refers to the set of invertible $n \times n$ matrices.

Proposition 18.10 (Exercise 1.4.8a)**.** *Let G be a subgroup of $\mathrm{GL}(n, \mathbb{R})$ and denote the identity (matrix) by I. Let $L_g : G \to G$ be the map $L_g(h) = gh$. Then $(dL_g)_I : T_IG \to T_gG$ is given by $(dL_g)_I(A) = gA$.*

Proof. Let $h : (-\epsilon, \epsilon) \to G$ be a smooth curve with $h(0) = I$ and $h'(0) = A \in M(n, \mathbb{R})$. Then

$$(dL_g)_I(A) = \frac{d}{dt}(L_g \circ h)(t)|_{t=0} = \frac{d}{dt}(gh(t))|_{t=0} = gh'(t)|_{t=0} = gh'(0) = gA$$

□

Corollary 18.11 (to Exercise 1.4.8a). *Define the vector field $\tilde{A}$ on G by $\tilde{A}(g) = gA$. Then $\tilde{A}$ is left-invariant.*

Proof.

$$\tilde{A}(gh) = ghA = g(hA) = g(\tilde{A}(h)) = (dL_g)_h(\tilde{A}(h))$$

□

19 Connection of $\mathrm{SL}(n, \mathbb{C})$ to $\mathrm{sl}(n, \mathbb{C})$

Definition 19.1. *Let $A \in gl(n, \mathbb{C})$. The* ***matrix exponential*** *is defined by*

$$\exp(A) = \sum_{n=1}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

Lemma 19.2. *For a matrix $A \in gl(n, \mathbb{C}), \det(\exp(A)) = e^{\operatorname{tr} A}$.*

Proposition 19.3. $\mathrm{sl}(n, \mathbb{C})$ *is the tangent space at the identity of* $\mathrm{SL}(n, \mathbb{C})$.

Proof. Note that

$$\mathrm{Sl}(n, \mathbb{C}) = \det{}^{-1}(1) \subset \mathrm{GL}(n, \mathbb{C})$$

We know that $\dim \mathrm{SL}(n, \mathbb{C}) = n^2 - 1$, so the tangent space at the identity must also have $\dim n^2 - 1$. Let $A \in \mathrm{sl}(n, \mathbb{C})$, so $\operatorname{tr} A = 0$. Let $\alpha : (-\epsilon, \epsilon) \to gl(n, \mathbb{C})$ be a curve with $\alpha(t) = \exp(tA)$. Then

$$\begin{aligned} \alpha(0) &= I \\ \alpha'(t) &= A \exp(tA) \implies \alpha'(0) = A \\ \det(\alpha(t)) &= \det(\exp(tA)) = e^{\operatorname{tr} A} = e^0 = 1 \end{aligned}$$

Thus α is a curve in $\mathrm{SL}(n, \mathbb{C})$ with $A = \alpha'(0)$ in the tangent space, thus $\mathrm{sl}(n, \mathbb{C})$ is contained in the tangent space. Since $\dim \mathrm{sl}(n, \mathbb{C}) = n^2 - 1$, it must be the whole tangent space. □

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